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Anatoliy M Samoilenko
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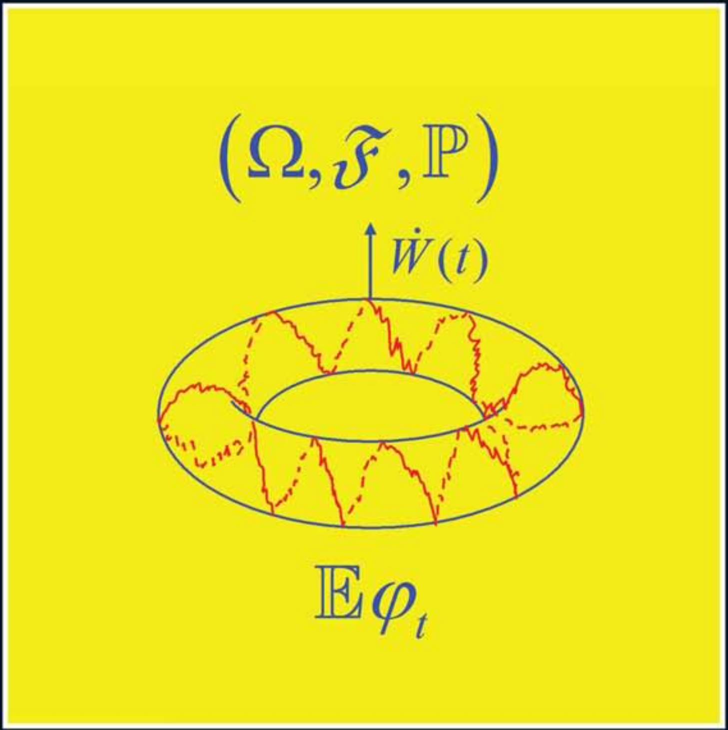

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Contents

Introduction

Differential equations with random perturbations constitute a mathematical model for real processes that are not described by deterministic laws, and evolve depending on random factors. Their modern theory lies at the junction of two areas of mathematics — random processes and ordinary differential equations. For this reason, the methods that are used to study the theory come both from the theories of random processes and differential equations.

In this book, we study stochastic equations from the point of view of ordinary differential equations, developing both asymptotic and qualitative methods that originated in the classical theory of differential equations and celestial mechanics. Here, the probability specifics make a direct application of the methods of classical theory more difficult and sometimes impossible. For example, the condition that a solution of the stochastic equation be adapted does not permit its continuation in both ways, but this property allows to apply many methods of this theory of ordinary differential equations. The additional term in the formula for stochastic derivative of a composition of functions often impedes the use of nonlinear change of variables so as to reduce it to an equation of a special form. It is well known that a change of variables is widely used in the theory of ordinary differential equations. The same probabilistic nature impacts qualitative properties of solutions. Results that are obtained in the theory are often similar to those in the classical theory of differential equations, yet often differ from them in an essential way. For example, the existence of a periodic solution of an equation with random periodic perturbations is equivalent to the existence of a bounded solution of the equation. This is an effect that is observed in the theory of ordinary differential equations only in the linear case. There is a very large class of ordinary differential equations that have solutions that are stable but not asymptotically stable. Such a situation is rather an exception than a rule for stochastic systems — it is almost always the case that stability of a solution implies its asymptotic stability. These examples show that equations with random perturbations resemble ordinary differential equations more in form than in essence. Studying them with methods used in the theory of ordinary differential equations requires a significant rethinking and extension of these methods, which constitute the subject of the monograph.

It is worth mentioning that there are many works that deal with stochastic differential equations in finite dimensional spaces. Here, for example, in the monographs of Vatanabe and Ikeda [195], Gikhman and Skorokhod [51], Skorokhod [54], Portenko [128], the main attention is paid to probability problems that relate to both the solutions and the stochastic processes. The ideology of this book is similar to that of the monographs of Ventsel' and Freidlin [192], Skorokhod [156], Arnold [8], Tsar'kov [186], Tsar'kov and Yasinskii [187], and especially to the work of Khas'minskii [70], where the authors examine problems that are natural for classical differential equations. Naturally, since that time the latter work had appeared, the classical theory of differential equations itself had developed significantly. And there appeared new results, new problems, and new methods to solve them. The aim of this book is to precisely develop these methods for equations with random perturbations. Together with this, the questions considered in this book are directly related to the interests of one of this book's authors, who studied similar problems for ordinary differential equations.

The main topics this book will address are:

- Systems of differential equations with the right-hand sides perturbed with random processes that have sufficiently regular trajectories; we call them systems with regular random perturbations.

- Differential equations with random right-hand sides and random impulsive effects.

- Systems of stochastic Ito equations.

The monograph contains five chapters. The first one deals with qualitative behavior of solutions of equations with random impulses. In particular, here we obtain conditions for dissipativity, stability of solutions of such systems, and study solutions that are periodic in the strict sense. For periodic systems, we study linear and weakly nonlinear cases in detail.

In Chapter 2, we develop an integral manifold method for differential systems with regular random perturbations, and Ito systems. We obtain conditions for the existence of invariant sets and for their stability. We also conduct a study of the behavior of invariant sets if the right-hand sides undergo small perturbations. An analogue of reduction principle in the stability theory is obtained for stochastic systems.

In Chapter 3, we study asymptotic behavior of solutions at infinity, in particular, the dichotomy and asymptotic equivalence between stochastic and deterministic systems. Linear and quasilinear stochastic Ito systems are also considered. For linear and weakly nonlinear stochastic systems, we find conditions for the existence of solutions that are mean square bounded on the axis and of solutions that are periodic in the strict sense.

Chapter 4 deals with extensions of stochastic systems on a torus. These extensions can be regarded as mathematical models for oscillating processes with random perturbations. A stochastic analogue of Green's function is introduced for the problem on invariant torus. We use this to obtain a representation for the random invariant torus in terms of a stochastic integral. This permits us to find conditions for the existence of invariant tori for both the linear and the nonlinear stochastic extensions of dynamical systems on a torus.

An application of an asymptotic averaging method to equations with random perturbations is a subject of Chapter 5. Here we obtain analogues of M. M. Bogolyubov's classical theorems for an averaging method for systems with random impulsive perturbations and stochastic Ito systems. The obtained results are used to study oscillating systems with small random perturbations.

The chapters are subdivided into sections. Formulas, lemmas, and theorems are numbered within each chapter with a double index. For example, formula (10) in Chapter 1 is indexed with (1.10). The same notation is used when referring to it in other chapters.

The reader is expected to know the basics of the qualitative theory of differential equations and the theory of random processes. We thus use the main standard notions and facts of these theories without further definitions or explanations.

The results that are given in the monograph are obtained by the authors with an exception of those discussed in Chapter 3, Sections 3.5–3.7. There, we make corresponding references to the original works.

This book is intended for mathematicians who work in the areas of asymptotic and qualitative analysis of differential equations with random perturbations. We also hope that it will be useful to specialists interested in oscillating processes subject to random factors.

The authors express their sincere gratitude to M. I. Portenko, Corresponding member of the National Academy of Sciences of Ukraine, for reading the manuscript and making valuable comments and advice that have led to an improvement of the monograph.

Chapter 1

Differential equations with random right-hand sides and impulsive effects

In this chapter, we study impulsive differential systems with random right-hand sides and random impulsive effects under the condition that their solutions do not possess the Markov property. Here we will be mainly dealing with problems pertaining to the qualitative theory of differential equations. In particular, we consider conditions for the existence of solutions that are bounded in probability, study conditions for such systems to be dissipative, consider stability of these systems. We also study conditions that would yield the existence of periodic solutions of systems with impulsive effects.

In Section 1.1, we give a definition of a solution of differential equation with random impulsive effect and prove a theorem about existence of the solution and its uniqueness.

Section 1.2 deals with finding conditions for an impulsive system to be dissipative. We obtain such conditions in terms of Lyapunov functions for the unperturbed portion of the impulsive system. This makes such conditions easy to verify.

Stability of the trivial solution in various probability system is studied in Section 1.3. The main method used there is the Lyapunov function method applied to the impulsive system with a random term being added. Thus, conditions for stability of the impulsive system with random perturbation are obtained in terms of conditions imposed on an unperturbed deterministic system.

Stability of the trivial solution of an unperturbed system, which is influenced by constantly acting random perturbations of both the continuous and the impulsive portions of the system, is studied in Section 1.4. Here, as opposed to the case of a deterministic system, asymptotic stability of the trivial solution of the unperturbed system is not sufficient for the perturbed system to be stable — it is necessary for the zero solution to be exponentially stable.

Sections 1.5–1.6 deal with periodically distributed solutions of essentially nonlinear impulsive systems with periodic right-hand side and periodic impulsive effect. We show here that a necessary and sufficient condition for existence of a periodic solution is existence of a solution that is probability bounded.

Linear and weakly nonlinear periodic impulsive systems are studied in Sections 1.7 and 1.8 using the Green's function method applied to the linear deterministic impulsive part of the system.

1.1 An impulsive process as a solution of an impulsive system

Let $(\Omega, F, \mathbf{P})$ be a complete probability space, $\xi(t)$ $(t \in \mathbf{R})$ a measurable, separable random process on the probability space, and η_i a sequence of random variables that take values in the spaces $\mathbf{R}^k$ and $\mathbf{R}^l$, correspondingly.

Consider a differential system with random right-hand side and a random impulsive effect at fixed times t_i,

$$\begin{aligned} \frac{dx}{dt} &= f(t, x, \xi(t)), \qquad t \neq t_i, \\ \Delta x|_{t=t_i} &= x(t_i + 0) - x(t_i - 0) = I_i(x, \eta_i), \end{aligned} \tag{1.1}$$

where $t \in \mathbf{R},\ x \in \mathbf{R}^n, i \in Z,\ f : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n,\ I_i : \mathbf{R}^n \times \mathbf{R}^l \to \mathbf{R}^n$, $\{t_i\}$ is a given number sequence.

A solution of system (1.1) is a random process $x(t, \omega)$ that is piecewise absolutely continuous on the intervals $(t_i, t_{i+1}]$, left-continuous at points t_i with probability 1, and satisfies, with probability 1, the first relation in (1.1) on the intervals $(t_i, t_{i+1}]$, and the second relation in (1.1) for $t = t_i$ (the jump conditions). We will also assume that the sequence of times of the impulsive effects does not have finite limit points.

In what follows, we will need conditions implying that a solution of the Cauchy problem for system (1.1) exists and can be continued into both directions. We give one such theorem.

Theorem 1.1. *Let the functions $f(t,x,y), I_i(x,z)$ be measurable in the totality of their variables, and the following conditions be satisfied.*

1. *There exists a random process $B(t)$, locally integrable on $\mathbf{R}$, and a sequence of random variables $L_i(\omega)$ such that*

$$|f(t,x_1,\xi(t)) - f(t,x_2,\xi(t))| \le B(t)|x_1 - x_2|,$$
$$|I_i(x_1,\eta_i) - I_i(x_2,\eta_i)| \le L_i|x_1 - x_2|,$$

   for arbitrary $x_1, x_2 \in \mathbf{R}^n$.

2. *For arbitrary $T \in \mathbf{R}$,*

$$\mathbf{P}\Big\{\int_0^T |f(t,0,\xi(t))|dt < \infty\Big\} = 1.$$

3. *The mappings $A_i : A_i x = x + I_i(x,z)$ map the space $\mathbf{R}^n$ bijectively onto $\mathbf{R}^n$ for each $z \in \mathbf{R}^l$.*

Then there exists a solution $x(t,\tau,x_0(\omega))$ of the Cauchy problem for system (1.1) *with the initial condition $x(\tau,\tau,x_0(\omega)) = x_0(\omega)$, it is unique trajectory-wise, and makes a piecewise absolutely continuous random process for all $t \in \mathbf{R}$.*

A proof of this theorem can be easily obtained from the definition of a solution of system (1.1), corresponding existence and uniqueness theorems for systems with random right-hand sides without impulses [70, p. 26], and such theorems for ordinary differential systems with impulsive effects [143, pp. 7–12].

Remark. If we will be interested in continuing the solution to the right of the initial point τ, then condition 3 in Theorem 1.1 can be weakened and replaced with the following condition:

— the mapping $A_i x$ is defined on the whole space $\mathbf{R}^n$ for every $z \in \mathbf{R}^l$.

If not stated otherwise, everywhere below the conditions for existence and uniqueness of a solution of the Cauchy problem for system (1.1) will mean precisely the conditions of Theorem 1.1.

1.2 Dissipativity

In this section, we will study conditions that would imply that solutions of system (1.1) are bounded and dissipative in probability.

Let us make definitions needed in the sequel.

Definition 1.1. A random process $\zeta(t)$ $(t \geq 0)$ will be called *bounded in probability* if the random variables $|\zeta(t,\omega)|$ are bounded in probability with respect to t, that is,

$$\sup_{t>0} \mathbf{P}\{|\zeta(t)| > R\} \to 0, \qquad R \to \infty .$$

A random variable $x_0(\omega)$ belongs to class A_{R_0} if

$$\mathbf{P}\{|x_0(\omega)| < R_0\} = 1 . \tag{1.2}$$

Definition 1.2. System (1.1) is called *dissipative* if all its solutions can be unboundedly continued to the right and the random variables $|x(t,w,x_0,t_0)|$ are bounded in probability uniformly with respect to $t \geq t_0$ and $x_0(\omega) \in A_{R_0}$ for any R_0.

Naturally, it is difficult to get constructive dissipativity conditions without imposing some conditions on system (1.1). However, for special systems where the random process and the random variables enter linearly, one can give effective enough dissipativity conditions in terms of Lyapunov functions.

In the sequel, we will assume that all Lyapunov functions $V(t,x)$ are absolutely continuous in t, uniformly continuous in x in a neighborhood of every point, and Lipschitz continuous with respect to x,

$$|V(t,x_1) - V(t,x_2)| \leq B|x_1 - x_2| ,$$

in the domain $|x| \leq R$, $t \in [0,T]$, where B depends on R and T. In this case, we will say that $V \in C$. If the constant B is independent of the domain, we will denote it by $V \in C_0$.

It is clear that, if $V \in C$ and $x(t)$ is absolutely continuous, then $V(t,x(t))$ is also absolutely continuous and the Laplace operator defined by the formula

$$\frac{d^0 V(t,x)}{dt} = \overline{\lim_{h\to 0}} \frac{V(t+h, x(t+h,t,x)) - V(t,x)}{h}$$

will become

$$\frac{d^0 V}{dt} = \frac{d}{dt} V(t,x(t))|_{x(t)=x}$$

in this case.

In the sequel, we will need the following result on linear inequalities [70].

Lemma 1.1. *Let $y(t)$ be an absolutely continuous function for $t \geq t_0$ such that its derivative, $\frac{dy}{dt}$, satisfies the inequality*

$$\frac{dy}{dt} \leq A(t)y + B(t)$$

for almost all $t \geq t_0$, where $A(t)$, $B(t)$ are continuous almost everywhere and integrable on every bounded interval $t \geq t_0$.

Then

$$y(t) \leq y(t_0)\ \exp\Big\{\int_{t_0}^{t} A(s)ds\Big\} + \int_{t_0}^{t} \exp\Big\{\int_{s}^{t} A(u)du\Big\} B(s)\, ds$$

for almost all $t \geq t_0$.

Consider now an impulsive system in a special form,

$$\begin{aligned} \frac{dx}{dt} &= F(t,x) + \sigma(t,x)\xi(t), \qquad t \neq t_i, \\ \Delta x|_{t=t_i} &= I_i(x) + J_i(x)\eta_i(\omega), \quad\ t = t_i\,. \end{aligned} \tag{1.3}$$

Here the functions F, I_i and the matrices σ, J_i of dimension $n \times d$ are defined and continuous in the totality of the variables for $t \geq 0$, $x \in \mathbf{R}^n$, F, and σ is locally Lipschitz in x, $\xi(t)$ is a measurable separable random process, η_i is a sequence of random variables, $\xi(t)$ and η_i take values in $\mathbf{R}^d$.

Together with (1.3), consider the truncated deterministic system

$$\begin{cases} \dfrac{dx}{dt} = F(t,x), \qquad t \neq t_i, \\ \Delta x|_{t=t_i} = I_i(x), \quad\ t = t_i\,. \end{cases} \tag{1.4}$$

Denote by $\frac{d^{(1)}}{dt}$ the Lyapunov operator for (1.3), and by $\frac{d^{(0)}}{dt}$ the Lyapunov operator for the truncated system (1.4).

Lemma 1.2. *If $V(t,x) \in C_0$, then*

$$\begin{aligned} &\frac{d^{(1)}V(t,x)}{dt} \leq \frac{d^{(0)}V(t,x)}{dt} + B\|\sigma(t,x)\||\xi(t)|, \qquad t \neq t_i\,, \\ &V(t_i, x + I_i(x) + J_i(x)\eta_i(\omega)) \\ &\qquad\qquad \leq V(t_i, x + I_i(x)) + B|\eta_i|\|J_i(x)\|\,, \qquad t = t_i\,, \end{aligned} \tag{1.5}$$

for almost all t with probability 1.

Proof. The first inequality in (1.5) follows from a corresponding fact in [70, p. 28]. The second one is obtained from

$$\begin{aligned} &V(t_i, x + I_i(x) + J_i(x)\eta_i(\omega)) \\ &\quad = V(t_i, x + I_i(x) + J_i(x)\eta_i) - V(t_i, x + I_i(x)) + +V(t_i, x + I_i(x)) \\ &\quad \leq V(t_i, x + I_i(x)) + B|\eta_i|\|J_i(x)\|\,. \end{aligned}$$

□

Conditions for system (1.3) to be dissipative will be given in terms of Lyapunov functions for system (1.4), as it was done in [71].

The following theorem is a main result of this subsection.

Theorem 1.2. *Let in the domain $x \in \mathbf{R}^n$, $t \geq 0$, there exist a nonnegative Lyapunov function $V(t,x) \in C_0$ satisfying the condition*

$$V_R = \inf_{|x|>R,\, t>t_0} V(t,x) \to \infty, \qquad R \to \infty\,, \tag{1.6}$$

and also

$$\frac{d^0 V}{dt} \leq -C_1 V,\ V(t_i, x + I_i(x)) - V(t_i, x) \leq -C_2 V(t_i, x),$$

where C_1, C_2 are positive constants.

Also assume that F and σ are locally Lipschitz continuous in x and $\|J_i(x)\| \leq C_3$, $\| \sigma(t,x) \| \leq C_4$, where $C_3 > 0$ and $C_4 > 0$ are constants.

Then system (1.3) *is dissipative for any measurable separable random process $\xi(t)$ and any sequence of random variables $\{\eta_i\}$ such that*

$$\sup_{t\geq 0} \mathbf{E}|\xi(t,w)| < \infty, \qquad \sup_{i\geq 0} \mathbf{E}|\eta_i(w)| < \infty.$$

Before proving the theorem, we need a lemma from [70, p. 32] which is a form of Chebyshev's inequality that will be convenient for subsequent calculations.

Lemma 1.3. *Let $V(t,x)$ be a nonnegative function, and $\eta(t)$ a random process such that $\mathbf{E}V(t,\eta(t))$ exists. Then*

$$\mathbf{P}\big\{|\eta(t)| > R\big\} \leq \frac{\mathbf{E}V(t,\eta(t))}{\inf_{|x|>R,\, s>t_0} V(s,x)}\,. \tag{1.7}$$

Proof of Theorem 1.2. First of all, it easily follows from the conditions of the theorem that a solution of system (1.3) can be infinitely continued to the right. Now, without loss of generality, we can assume that $t_0 = 0$. Let $x(t,x_0)$ be a solution of the Cauchy problem for (1.3) such that $x(0,x_0) = x_0(w)$. Then, for $t \in [0,t_1]$ where t_1 is the time of the first impulse, Lemma 1.2 implies that

$$\begin{aligned}\frac{dV(t,x(t,x_0))}{dt} &\leq \frac{d^{(0)}V(t,x(t,x_0))}{dt} + B\|\sigma(t,x(t,x_0))\|\,|\xi(t)| \\ &\leq -C_1 V(t,x(t,x_0)) + BC_4|\xi(t)|\,.\end{aligned}$$

It follows from Lemma 1.1 that

$$V(t, x(t, x_0)) \leq V(0, x_0)e^{-c_1 t} + BC_4 \int_0^t e^{C_1(s-t)}|\xi(s)|ds\,, \tag{1.8}$$

where B is a Lipschitz constant for V.

Taking expectation from both sides of (1.8) we get

$$\mathbf{E}V(t, x(t, x_0)) \leq Ke^{-C_1 t} + N_1\,,$$

where

$$K = \sup_{x_o \in A_{R_0}} V(0, x_0), \qquad N_1 = \frac{BC_4}{C_1} \sup_{t \geq 0} \mathbf{E}|\xi(t)|\,.$$

The action of impulses gives

$$\begin{aligned} V(t_1+0, x(t_1+0)) &= V(t_1, x(t_1) + I_1(x(t_1)) + J_1(x(t_1))\eta_1(\omega)) \\ &\leq V(t_1, x(t_1) + I_1(x(t_1)) + B|\eta_1(w) \parallel J_1(x(t_1) \parallel \\ &\leq (1 - C_2)V(t_1, x(t_1)) + BC_3|\eta_1(w)| \end{aligned} \tag{1.9}$$

and, hence,

$$\mathbf{E}V(t_1+0, x(t_1+0)) \leq (1 - C_2)(Ke^{-C_1 t_1} + N_1) + BC_3\mathbf{E}|\eta_1(w)|\,. \tag{1.10}$$

Denote $N_2 = BC_3 \sup_{i \geq 0} \mathbf{E}|\eta_i|$. Then

$$\mathbf{E}V(t_1+0, x(t_1+0)) \leq (1 - C_2)(Ke^{-C_1 t_1} + N_1) + N_2\,. \tag{1.11}$$

Let us now consider the behavior of the solution on the interval $(t_1, t_2]$. For $t \in (t_1, t_2]$, we have

$$V(t, x(t, x_0)) \leq e^{-C_1(t-t_1)}V(t_1+0, x(t_1+0, x_0)) + BC_4 \int_{t_1}^t e^{C_1(s-t)}|\xi(s)|ds\,. \tag{1.12}$$

Taking expectation of both sides and using (1.12) we get

$$\begin{aligned} \mathbf{E}V(t, x(t, x_0)) &\leq e^{-C_1(t-t_1)}((1 - C_2)(Ke^{-C_1 t_1} + N_1) + N_2) + N_1 \\ &\leq e^{-C_1 t}(1 - C_2)K + e^{-C_1(t-t_1)}N_2 + e^{-C_1(t-t_1)}(1 - C_2)N_1 + N_1\,. \end{aligned} \tag{1.13}$$

It is clear that

$$V(t_2+0, x(t_2+0)) \leq (1 - C_2)V(t_2, x(t_2, x_0)) + BC_3|\eta_2(w)|$$

and, hence,

$$\mathbf{E}V(t_2+0, x(t_2+0)) \le (1-C_2)(e^{-C_1 t_2}(1-C_2)K + e^{-C_1(t_2-t_1)}[N_1+N_2]+N_1)+N_2\,.$$

Thus, we have the following estimate for $t \in (t_2, t_3]$:

$$V(t, x(t, x_0)) \le e^{-C_1(t-t_2)}((1-C_2)V(t_2, x(t_2, x_0)) + BC_3|\eta_2(w)|) + BC_4 \int_{t_2}^{t} e^{C_1(s-t)}|\xi(s)|ds\,.$$

Taking expectation of both sides we have

$$\mathbf{E}V(t, x(t, x_0)) \le (1-C_2)^2 e^{-C_1 t}K + (1-C_2)e^{-C_1(t-t_1)}[N_1+N_2] + e^{-C_1(t-t_2)}[N_1+N_2] + N_1\,.$$

Similar considerations show that the quantity $\mathbf{E}V(t, x(t, x_0))$ is uniformly bounded in $t \ge 0$ and $x_0(\omega) \in A_{R_0}$, since it is majorized by

$$[N_1+N_2]((1-C_3)^n + (1-C_3)^{n-1} + \cdots + 1) + N + K,$$

for $t \in (t_n,\, t_{n+1}]$, where the parentheses contain a decreasing geometric progression with the ratio $(1-C_2) < 1$. So, inequality (1.7) shows that

$$\mathbf{P}\big\{|x(t, x_0)| > R\big\} \le \frac{K_1}{\inf_{|x|>R,\, s>0} V(s, x)}\,.$$

The proof of the theorem ends with a use of condition (1.16). □

We now give conditions that would imply that solutions of system (1.3) are not only bounded in probability, but also have bounded moments. We will impose stricter conditions on the Lyapunov function. Assume that

$$V(t, x) > C_7|x| - C_8\,, \tag{1.14}$$

where C_7 and C_8 are positive constants.

Theorem 1.3. *Let the functions V, F, σ, I_i, J_i satisfy the conditions of Theorem 1.2 with the constant $C_2 > \frac{1}{2}$, and let inequality* (1.14) *hold. Also assume that*

$$\sup_{t\ge 0} \mathbf{E}|\xi(t)|^\alpha < \infty, \ \sup_{i\in N} \mathbf{E}|\eta_i(w)|^\alpha < \infty$$

for some $\alpha > 1$. Then a solution $x(t, x_0)$ of the Cauchy problem for system (1.3) *satisfies the estimate*

$$\sup_{t\ge 0} \mathbf{E}|x(t, w)|^\alpha < \infty\,. \tag{1.15}$$

Proof. Consider the Lyapunov function $W(t,x) = [V(t,x)]^\alpha$. Then

$$\begin{aligned}\frac{dW(t,x(t,x_0))}{dt} &= \alpha V^{\alpha-1}(t,x(t,x_0))\frac{dV(t,x(t,x_0))}{dt}\\ &\le -C_9 W(t,x(t,x_0)) + C_{10}V^{\alpha-1}(t,x(t,x_0))|\xi(t,w)|\,. \end{aligned} \tag{1.16}$$

The other term in (1.16) is estimated using Young's inequality,

$$V^{\alpha-1}(t,x(t,x_0))|\xi(t,w)| \le \frac{V^\alpha(t,x(t,x_0))}{\alpha}(\alpha-1) + \frac{|\xi(t)|^\alpha}{\alpha}\,. \tag{1.17}$$

Inequality (1.17) gives the following for $W(t,x(t,x_0))$:

$$\frac{dW(t,x(t,x_0))}{dt} \le -C_{11}W(t,x(t,x_0)) + C_{12}|\xi(t)|^\alpha\,. \tag{1.18}$$

So, as in the previous theorem,

$$W(t,x(t,x_0)) \le e^{-C_{11}t}W(0,x_0) + C_{12}\int_0^t e^{-C_{11}(s-t)}|\xi(s)|^\alpha ds \tag{1.19}$$

on $(0,t_1]$. And, hence,

$$\mathbf{E}W(t,x(t,x_0)) \le K_2 e^{-C_{11}t} + N_3\,, \tag{1.20}$$

where $K_2 = \sup_{x_0\in A_{R_0}} W(0,x_0)$ and $N_3 = \frac{C_{12}}{C_{11}}\sup_{t\ge 0}\mathbf{E}|\xi(s)|^\alpha$. Then

$$\begin{aligned}W(t_1+0,x(t_1+0,x_0)) &= [V(t_1+0,x(t_1+0,x_0))]^\alpha\\ &\le ((1-C_2)V(t_1,x(t_1,x_0)) + BC_3|\eta_1|)^\alpha\\ &\le 2^{\alpha-1}(1-C_2)^\alpha V^\alpha(t_1,x(t_1,x_0)) + 2^{\alpha-1}(BC_3)^\alpha|\eta_1|^\alpha\,.\end{aligned}$$

Hence,

$$W(t_1+0,x(t_1+0,x_0)) \le 2^{\alpha-1}(1-C_2)^\alpha W(t_1,x(t_1,x_0)) + 2^{\alpha-1}(BC_3)^\alpha|\eta_1|^\alpha\,. \tag{1.21}$$

It follows from (1.19) and (1.20) that

$$\begin{aligned}W(t,x(t,x_0)) \le{}& e^{-C_{11}(t-t_1)}\left[2^{\alpha-1}(1-C_2)^\alpha W(t_1,x(t_1,x_0)) + 2^{\alpha-1}(BC_3)^\alpha|\eta_1|^\alpha\right]\\ &+ C_{12}\int_{t_1}^t e^{C_{11}(s-t)}|\xi(s)|^\alpha ds\end{aligned}$$

for $t \in (t_1,t_2]$.

Taking expectation of both sides of the inequality we get

$$\begin{aligned}\mathbf{E}W(t,x(t,x_0)) &\le e^{-C_{11}(t-t_1)}[2^{\alpha-1}(1-C_2)^\alpha(e^{-C_{11}t_1}\mathbf{E}W(0,x_0)+N_3)\\ &\quad + 2^{\alpha-1}(BC_3)^\alpha \sup_{i\in N}\mathbf{E}|\eta_1|^\alpha] + N_3\\ &\le 2^{\alpha-1}(1-C_2)^\alpha e^{-C_{11}t}K_2 + e^{-C_{11}(t-t_1)}[N_3+N_4]+N_3\,,\end{aligned} \tag{1.22}$$

where $N_4 = 2^{\alpha-1}(BC_3)^\alpha \sup_{i\in N}\mathbf{E}|\eta_1|^\alpha$. But

$$W(t_2+0,x(t_2+0,x_0)) \le 2^{\alpha-1}(1-C_2)^\alpha W(t_2,x(t_2,x_0)) + 2^{\alpha-1}(BC_3)^\alpha|\eta_2|^\alpha\,.$$

We have the following estimate on the interval $t\in(t_2,t_3]$:

$$\begin{aligned}W(t,x(t,x_0)) &\le e^{-C_{11}(t-t_2)}[2^{\alpha-1}(1-C_2)^\alpha e^{-C_{11}t_2}2^{\alpha-1}(1-C_2)^\alpha K_2\\ &\quad +2^{\alpha-1}(1-C_2)^\alpha(e^{-C_{11}(t_2-t_1)}[N_3+N_4]+N_3)]+N_3\\ &= e^{-C_{11}t}2^{2(\alpha-1)}(1-C_2)^{2\alpha}K_2\\ &\quad + e^{-C_{11}(t-t_1)}2^{\alpha-1}(1-C_2)^\alpha[N_3+N_4]\\ &\quad + N_3 2^{\alpha-1}(1-C_2)^\alpha e^{-C_{11}(t-t_2)} + N_3\,.\end{aligned} \tag{1.23}$$

Reasoning as in the proof of Theorem 1.2 we see that $\mathbf{E}W(t,x(t,x_0))$ is bounded uniformly in $t\ge 0$. A use of the inequality

$$W(t,x) > C_{13}|x|^\alpha - C_{14},$$

which follows from (1.14), ends the proof. □

1.3 Stability and Lyapunov functions

The results in this section are based on [172], where the authors studied stability of solutions of system (1.1). Without loss of generality, we assume that the system has a trivial solution, stability of which will be studied. Similarly to systems with random right-hand sides without impulsive effects, stability here can be understood in different senses. We give needed definitions.

Definition 1.3. Let $x(t)\equiv 0$ be a solution of system (1.1).

1. The solution is called *stable in probability* (for $t\ge t_0$) if for arbitrary $\varepsilon>0$ and $\delta>0$ there exists $r>0$ such that

$$\mathbf{P}\{|x(t,\omega,t_0,x_0)|>\varepsilon\}\le\delta \tag{1.24}$$

for $t\ge t_0,\ |x_0|<r$.

2. The solution is called *asymptotically stable in probability* if it is stable with probability 1 and for an arbitrary $\varepsilon > 0$ there exists $r(\varepsilon)$ such that

$$\mathbf{P}\left\{|x(t,\omega,t_0,x_0)| > \varepsilon\right\} \to 0$$

as $t \to \infty$ if $|x_0| < r$.

3. The solution is *p-stable* if for arbitrary $\varepsilon > 0$ there exists $r > 0$ such that

$$\mathbf{E}|x(t,\omega,t_0,x_0)|^p < \varepsilon, \qquad p > 0,$$

for $t \geq t_0$ and $|x_0| < r$.

4. The solution is called *asymptotically p-stable* if it is p-stable and, for sufficiently small $|x_0|$,

$$\mathbf{E}|x(t,\omega,t_0,x_0)|^p \to 0$$

as $t \to \infty$.

5. The solution is called *totally stable in probability* if it is stable in probability and also for arbitrary x_0, $\varepsilon > 0$, $\delta > 0$ there exists $T = T(x_0,\varepsilon,\delta)$ such that inequality (1.24) holds for $t > T$. *Total asymptotic stability* and *total p-stability* is defined similarly.

6. The solution is called *exponentially p-stable* if there exist constants $A > 0$, $\alpha > 0$ such that

$$\mathbf{E}|x(t,\omega,t_0,x_0)|^p \leq A|x_0|^p \exp\{-\alpha(t-t_0)\}\,.$$

7. The solution is called *stable with probability* 1 in one or another sense if all trajectories are stable in that sense.

Naturally, it is difficult to obtain positive results about stability of the zero solution of system (1.1) if no additional conditions are imposed on the right-hand side of the system or on the random effects, e.g., the assumption that the random effects are Markov was made in [179]. We will again be limited to the case when the random component enters linearly to both the right-hand side of the system and the magnitude of the impulses, and the system itself has the form

$$\begin{aligned} \frac{dx}{dt} &= F(t,x) + \sigma(t,x)\xi(t), \qquad t \neq t_i, \\ \Delta x|_{t=t_i} &= x(t_i+0) - x(t_i-0) = I_i(x) + J(x)\eta_i(\omega)\,, \end{aligned} \tag{1.25}$$

where $t \geq 0$, $x \in \mathbf{R}^n$, $i \in Z$. The functions F, σ, I_i, J_i are defined and continuous in t, x, (F and I_i are vectors in $\mathbf{R}^n$, σ and J_i are $n \times l$-matrices), $\xi(t)$ is an l-dimensional random process locally integrable on any bounded line segment, η_i are l-dimensional random variables. We also assume that the functions F and σ are Lipschitz continuous with respect to $x \in \mathbf{R}^n$, and the sequence t_i has no finite limit points. Since we will be studying stability of the trivial solution, we assume that

$$F(t,0) \equiv 0, \qquad \sigma(t,0) \equiv 0, \qquad I_i(0) \equiv 0, \qquad J_i(0) \equiv 0\,.$$

Then conditions for stability of system (1.25) can be given in terms of Lyapunov functions of the truncated deterministic system

$$\begin{aligned} \frac{dx}{dt} &= F(t,x), \qquad t \neq t_i\,, \\ \Delta x|_{t=t_i} &= I_i(x)\,, \end{aligned} \tag{1.26}$$

in the way this was done in [70, p. 44] for a system without impulses.

Let us also remark that stability of a system of type (1.25) with Markov coefficients and impulsive effects in random time moments was studied using Lyapunov function in [65]. The authors have also indicated there principal difficulties in the case where the times of the impulsive effects are deterministic. This is related to the fact that the trajectories of solutions in this case are discontinuous with probability 1, as opposed to the case where the effects occur at random times.

Theorem 1.4. *Let there exist a function $V(t,x)$ on the domain $x \in \mathbf{R}^n$, $t \geq t_0$, for some $t_0 \in \mathbf{R}$, that is absolutely continuous in t and globally Lipschitz continuous in x with a Lipschitz constant L, and let the following conditions be satisfied:*

1) $V(t,x)$ is positive definite uniformly in t, that is,

$$\inf_{|x|>r,t>t_0} V(t,x) = V_r > 0, \qquad r > 0;$$

2) there exist constants $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ such that

$$\begin{gathered} \frac{d^0V}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} F(t,x) \leq -C_1 V(t,x)\,, \\ ||\sigma(t,x)|| + ||J_i(x)|| \leq C_2 V(t,x)\,, \\ V(t_i, x + I_i(x)) - V(t_i,x) \leq -C_3 V(t_i,x)\,, \\ V(t,0) = 0\,, \end{gathered}$$

for arbitrary $t \geq t_0$, $x \in \mathbf{R}^n$.

Let the process $|\xi(t)|$ and the sequence $|\eta_i|$ satisfy the law of large numbers, that is, for arbitrary $\varepsilon > 0$ and $\delta > 0$ there exist $T > 0$ and natural N_0 such that

$$\mathbf{P}\left\{\left|\frac{1}{t}\int_0^t |\xi(s)|\,ds - \frac{1}{t}\int_0^t \mathbf{E}|\xi(s)|\,ds\right| > \delta\right\} \le \varepsilon\,,$$

$$\mathbf{P}\left\{\left|\frac{1}{n}\sum_{i=1}^{n} |\eta_i| - \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}|\eta_i|\right| > \delta\right\} \le \varepsilon\,,$$

for arbitrary $t > T,\ n > N_0$, and let

$$\sup_{t>0} \mathbf{E}|\xi(t)| < \frac{C_1}{LC_2}\,,$$

$$\sup_{i\in N} \mathbf{E}|\eta_i| < \frac{C_3}{LC_2}\,.$$

Then the trivial solution of system (1.25) *is totally asymptotically stable in probability. If $|\xi(t)|$ and $|\eta_i|$ satisfy the strong law of large numbers, then the same conditions are sufficient for the trivial solution to be totally asymptotically stable with probability* 1*.*

Proof. Without loss of generality, we assume that $t_0 = 0$. From a result in [70, p. 45], we have the following estimate on the interval $[0,\ t_1]$:

$$V(t, x(t)) \le V(0, x_0)\exp\left\{LC_2\left(\frac{1}{t}\int_0^t |\xi(s)|\,ds - \frac{C_1}{LC_2}\right)t\right\},$$

where $x(t)$, $x(0) = x_0$, is a solution of system (1.25).

However, since

$$V(t_i, x + I_i(x) + J_i(x)\eta_i) - V(t_i, x) \le -C_3 V(t_i, x) + L|\eta_i|\,||J_i(x)||\,, \quad (1.27)$$

we have that

$$\begin{aligned} V(t_i, x(t_i) + I_i(x(t_i)) + J_i(x(t_i))\eta_i) &\le L|\eta_i|\,||J_i(x(t_i))|| + (1 - C_3)V(t_i, x(t_i)) \\ &\le (1 - C_3 + L|\eta_i|C_2)V(t_i, x(t_i))\,. \end{aligned}$$

Hence,

$$V(t_1{+}0, x(t_1{+}0)) \le (1{-}C_3{+}L|\eta_1|C_2)V(0, x_1)\exp\left\{\int_0^{t_1} \big(LC_2|\xi(s)|{-}C_1\big)\,ds\right\}.$$

So, for arbitrary $t \in (t_1,\ t_2]$, we have the estimate

$$V(t, x(t)) \le (1 - C_3 + L|\eta_1|C_2)V(0, x_0)\exp\left\{\int_0^t \big(LC_2|\xi(s)| - C_1\big)\,ds\right\}.$$

Similar considerations yield

$$V(t,x(t)) \le \prod_{i=1}^{n}(1 - C_3 + L|\eta_i|C_2)V(0,x_0)\exp\left\{\int_0^t (LC_2|\xi(s)| - C_1)\,ds\right\}$$

for arbitrary $t \in (t_n, t_{n+1}]$, $n \ge 1$. Let us write the product in the latter inequality as

$$\begin{aligned}\prod_{i=1}^{n}(1 - C_3 + L|\eta_i|C_2) &= \prod_{i=1}^{n}\exp\{\ln(1 - C_3 + L|\eta_i|C_2)\}\\ &= \exp\left\{\sum_{i=1}^{n}\ln(1 - C_3 + L|\eta_i|C_2)\right\}\\ &\le \exp\left\{\sum_{i=1}^{n}(L|\eta_i|C_2 - C_3)\right\}.\end{aligned}$$

Thus we have the estimate

$$\begin{aligned}V(t,x(t)) \le V(0,x_0)&\exp\left\{LC_2\left(\frac{1}{t}\int_0^t |\xi(s)|\,ds - \frac{C_1}{LC_2}\right)t\right\}\\ &\times \exp\left\{LC_2\left(\frac{1}{n}\sum_{i=1}^{n}|\eta_i| - \frac{C_3}{C_2L}\right)n\right\} \qquad (1.28)\end{aligned}$$

on $(t_n, t_{n+1}]$.

Consider the quantity

$$\begin{aligned}\mathbf{P}\Bigg\{V(0,x_0)&\exp\left\{LC_2\left(\frac{1}{t}\int_0^t |\xi(s)|\,ds - \frac{C_1}{LC_2}\right)t\right\}\\ &\times \exp\left\{LC_2\left(\frac{1}{n}\sum_{i=1}^{n}|\eta_i| - \frac{C_3}{C_2L}\right)n\right\} > V_\delta\Bigg\}\end{aligned}$$

for arbitrary $t \ge 0$. Denote by y the following:

$$y = V(0,x_0)\exp\left\{LC_2\left(\frac{1}{t}\int_0^t |\xi(s)|\,ds - \frac{C_1}{LC_2}\right)t + LC_2\left(\frac{1}{n}\sum_{i=1}^{n}|\eta_i| - \frac{C_3}{C_2L}\right)n\right\}.$$

Then

$$\mathbf{P}\{y > V_\delta\} = \mathbf{P}\left\{(y > V_\delta) \cap \left(\frac{1}{n}\sum_{i=1}^{n}|\eta_i| - \frac{C_3}{C_2 L} \le 0)\right\}\right.$$
$$+ \mathbf{P}\left\{(y > V_\delta) \cap \left(\frac{1}{n}\sum_{i=1}^{n}|\eta_i| - \frac{C_3}{C_2 L} > 0\right)\right\}. \qquad (1.29)$$

However,

$$\mathbf{P}\left\{(y > V_\delta) \cap \left(\frac{1}{n}\sum_{i=1}^{n}|\eta_i| - \frac{C_3}{C_2 L} \le 0\right)\right\}$$
$$\le \mathbf{P}\left\{V(0, x_0)\exp\left\{LC_2\left(\frac{1}{t}\int_0^t |\xi(s)|\,ds - \frac{C_1}{LC_2}\right)t\right\} > V_\delta\right\}. \quad (1.30)$$

It follows from [70, p. 46] that the probability in the right-hand side does not exceed $\frac{\varepsilon}{2}$ for x_0 sufficiently small in the norm.

Let us estimate the second term in (1.29). Using the law of large numbers we see that there exists n_0 such that

$$\mathbf{P}\left\{\frac{1}{n}\sum_{i=1}^{n}|\eta_i| > \frac{C_3}{LC_2}\right\} < \frac{\varepsilon}{2}$$

for arbitrary $n \ge n_0$, Hence, we get

$$\mathbf{P}\left\{(y > V_\delta) \cap \left(\frac{1}{n}\sum_{i=1}^{n}|\eta_i| - \frac{C_3}{C_2 L}\right) > 0\right\} < \frac{\varepsilon}{2}$$

for arbitrary $t \ge t_{n_0}$, where t_{n_0} is the time of n_0-th impulse. Now, choose $A > 1$ such that

$$\mathbf{P}\left\{LC_2\int_0^{t_{n_0}} |\xi(s)|\,ds + \sum_{i=1}^{t_{n_0}}|\eta_i| > \ln A\right\} < \frac{\varepsilon}{2}$$

and r small enough so that $V(0, x_0)A < V_\delta$ for $|x_0| < r$. Denote

$$\frac{1}{n}\sum_{i=1}^{n}|\eta_i| - \frac{C_3}{LC_2} = l_n\,.$$

Then, for $t < t_{n_0}$, we have

$$
\begin{aligned}
&\mathbf{P}\left\{(y > V_\delta) \cap (l_n > 0\right\} \\
&\quad \leq \mathbf{P}\left\{(V(0, x_0) \exp\left\{LC_2\left(\int_0^{t_{n_0}} |\xi(s)|\, ds + \sum_{i=1}^{n_0} |\eta_i|\right)\right\} > V_\delta) \cap (l_n > 0)\right. \\
&\quad \left. \cap \left(LC_2\left(\int_0^{t_{n_0}} |\xi(s)|\, ds + \sum_{i=1}^{n_0} |\eta_i|\right) > \ln A\right)\right\} \\
&\quad + \mathbf{P}\left\{(V(0, x_0) \exp\left\{LC_2\left(\int_0^{t_{n_0}} |\xi(s)|\, ds + \sum_{i=1}^{n_0} |\eta_i|\right)\right\} > V_\delta) \cap (l_n > 0)\right. \\
&\quad \left. \cap \left(LC_2\left(\int_0^{t_{n_0}} |\xi(s)|\, ds + \sum_{i=1}^{n_0} |\eta_i|\right) \leq \ln A)\right\} \leq \frac{\varepsilon}{2}\,. \qquad (1.31)
\end{aligned}
$$

Substituting (1.30) and (1.31) into (1.29) and considering the cases $t \leq t_{n_0}$ and $t > t_{n_0}$ separately we get

$$
\mathbf{P}\left\{|x(t)| > \delta\right\} \leq \mathbf{P}\left\{V(t, x(t)) \geq V_\delta\right\} < \varepsilon
$$

for $|x_0| < r$ and arbitrary $t \geq 0$. Using the relations

$$
\mathbf{P}\left\{\frac{1}{t}\int_0^t |\xi(s)|\, ds > \frac{C_1}{LC_2}\right\} \to 0, \qquad t \to \infty,
$$

$$
\mathbf{P}\left\{\frac{1}{n}\sum_{i=1}^{n} |\eta_i| > \frac{C_1}{LC_2}\right\} \to 0, \qquad n \to \infty,
$$

and the above we get the fist claim of the theorem. The second one is proved similarly. $\square$

Inequality (1.28) obtained in the proof of Theorem 1.4 can be used to study p-stability of system (1.25).

Theorem 1.5. *Let system* (1.25) *have a Lyapunov function* $V(t, x)$ *satisfying the conditions of the previous theorem and the inequality*

$$
V(t, x) > C|x|, \; C > 0,
$$

and let the process $\xi(t)$ *and the sequence* η_i *be such that*

$$
\mathbf{E} \exp\left\{K_1\left(\int_0^t |\xi(s)|\, ds + \sum_{i=1}^{n} |\eta_i|\right)\right\} < \exp\{K_2(t+n)\} \qquad (1.32)
$$

for some positive K_1 *and* K_2*, where* $t > 0$*,* $n \in N$*, and the constants* C_1*,* C_2*,* K_1*,* K_2*,* L *satisfy*

$$LK_2C_2 \le K_1C'\,,$$

where $C' = \max\{C_1, C_3\}$*.*

Then the zero solution of system (1.25) *is p-stable for* $p \le \frac{K_1}{LC_2}$*.*

A proof of this theorem is entirely similar to the proof of the corresponding theorem in [70, p. 46] so we do not give it here.

As follows from the example in [70, p. 47], the condition that the process $|\xi(t)|$ satisfies the law of large numbers can not be dropped for a system without impulses. However, the impulsive effect in the system under consideration permits not to impose this condition on the process $|\xi(t)|$.

Theorem 1.6. *Let all the conditions of Theorem 1.4 be satisfied except for the law of large numbers, which is replaced with the following condition:*

– *for arbitrary positive* ε *and* δ *there exists* $T > 0$ *such that*

$$\mathbf{P}\left\{\left|\frac{1}{t}\int_0^t |\xi(s)|\,ds + \frac{1}{t}\sum_{i=1}^{n(t)}|\eta_i| - \frac{1}{t}\int_0^t \mathbf{E}|\xi(s)|\,ds - \frac{1}{t}\sum_{i=1}^{n(t)}\mathbf{E}|\eta_i|\right| > \delta\right\} < \varepsilon \tag{1.33}$$

for $t > T$*, where* $n(t)$ *is the number of impulses on the interval* $(0,\ t]$*.*

Then the first statement of Theorem 1.4 holds true. If the event in (1.33) *occurs with probability* 0*, then the second statement of Theorem 1.4 holds true.*

Proof. We have the following chain of inequalities

$$\varepsilon > \mathbf{P}\left\{\left|\frac{1}{t}\int_0^t |\xi(s)|\,ds + \frac{1}{t}\sum_{i=1}^{n(t)}|\eta_i| - \frac{1}{t}\int_0^t \mathbf{E}|\xi(s)|\,ds - \frac{1}{t}\sum_{i=1}^{n(t)}\mathbf{E}|\eta_i|\right| > \delta\right\}$$

$$> \mathbf{P}\left\{\frac{1}{t}\int_0^t |\xi(s)|\,ds + \frac{1}{t}\sum_{i=1}^{n(t)}|\eta_i| > \frac{C_1}{LC_2} + \frac{1}{t}\frac{C_3}{LC_2}n(t)\right\}$$

for sufficiently large $t \ge T$ and small δ.

Then, for an arbitrary $t \ge T$, we have

$$\mathbf{P}\left\{V(0,x_0)\exp\left\{\int_0^t (LC_2|\xi(s)| - C_1)\,ds\right\}\exp\left\{\sum_{i=1}^{n(t)}(LC_2|\eta_i| - C_3)\right\} > V_\delta\right\}$$

$$= \mathbf{P}\left\{V(0,x_0)\left\{LC_2\left[\left(\frac{1}{t}\int_0^t |\xi(s)|\,ds - \frac{C_1}{LC_2}\right)t\right.\right.\right.$$

$$\left.\left.\left. + \left(\frac{1}{n(t)}\sum_{i=1}^{n(t)}|\eta_i| - \frac{C_3}{LC_2}\right)n(t)\right]\right\} > V_\delta\right\}$$

$$= \mathbf{P}\left\{V(0,x_0)\exp\left\{LC_2\left[\left(\frac{1}{t}\int_0^t |\xi(s)|\,ds - \frac{C_1}{LC_2}\right)t\right.\right.\right.$$

$$\left.\left.\left. + \frac{n(t)}{t}\left(\frac{1}{n(t)}\sum_{i=1}^{n(t)}|\eta_i| - \frac{C_3}{LC_2}\right)t\right]\right\} > V_\delta\right\}$$

$$= \mathbf{P}\{(V(0,x_0)\exp\{LC_2K(t)t\} > V_\delta) \cap (K(t) > 0)\}$$

$$+ \mathbf{P}\{(V(0,x_0)\exp\{LC_2K(t)t\} > V_\delta) \cap (K(t) \le 0)\}\,,$$

where $K(t)$ denotes the quantity

$$K(t) = \frac{1}{t}\int_0^t |\xi(s)|\,ds - \frac{C_1}{LC_2} + \frac{n(t)}{t}\left(\frac{1}{n(t)}\sum_{i=1}^{n(t)}|\eta_i| - \frac{C_3}{LC_2}\right).$$

However,

$$\mathbf{P}\left\{K(t) > 0\right\} \le \mathbf{P}\left\{\frac{1}{t}\left(\int_0^t |\xi(s)|\,ds + \sum_{i=1}^{n}|\eta_i|\right) - \frac{n(t)}{t}\frac{C_3}{LC_2} - \frac{C_1}{LC_2} > 0\right\} < \varepsilon\,,$$

and

$$\mathbf{P}\left\{(V(0,x_0)\exp\{LC_2K(t)\} > V_\delta) \cap (K(t) \le 0)\right\} \le \mathbf{P}\left\{V(0,x_0) > V_\delta\right\} = 0$$

for sufficiently small x_0.

Choose now a number $A > 1$ sufficiently large such that

$$\mathbf{P}\left\{LC_2(\int_0^T |\xi(s|)\,ds + \sum_{i=1}^{n(T)}|\eta_i|) > \ln A\right\} < \varepsilon.$$

Then

$$\mathbf{P}\left\{V(0,x_0)\exp\left\{LC_2\int_0^t |\xi(s)|\,ds - C_1t + LC_2\sum_{i=1}^{n(t)}|\eta_i| - C_3n(t))\right\} > V_\delta\right\}$$

$$
\begin{aligned}
&\le \mathbf{P}\left\{V(0,x_0)\exp\left\{LC_2\left(\int_0^t |\xi(s)|\,ds + \sum_{i=1}^{n(t)} |\eta_i|\right)\right\} > V_\delta\right\}\\
&\le \mathbf{P}\left\{\left(V(0,x_0)\exp\left\{LC_2\left(\int_0^T |\xi(s)|\,ds + \sum_{i=1}^{n(T)} |\eta_i|\right)\right\} > V_\delta\right)\right.\\
&\left.\cap\left(LC_2\left(\int_0^T |\xi(s)|\,ds + \sum_{i=1}^{n(T)} |\eta_i|\right) > \ln A\right)\right\}\\
&\quad + \mathbf{P}\left\{\left(V(0,x_0)\exp\left\{LC_2\left(\int_0^T |\xi(s)|\,ds + \sum_{i=1}^{n(T)} |\eta_i|\right)\right\} > V_\delta\right)\right.\\
&\left.\cap\left(LC_2\left(\int_0^T |\xi(s)|\,ds + \sum_{i=1}^{n(T)} |\eta_i|\right) \le \ln A\right)\right\}\\
&\le \varepsilon + \mathbf{P}\left\{V(0,x_0)A > V_\delta\right\} = \varepsilon\,,
\end{aligned}
$$

which is true because of the choice of $|x_0| < r$ satisfying $V(0,x_0)A < V_\delta$. □

In the theory of deterministic differential systems with impulsive effects there are examples where an unstable differential system can be made stable by introducing impulsive effects, see [143].

Let us give an example showing that impulsive effects even at random times could turn an unstable system into an even asymptotically stable.

Example. Let $\tau(\omega)$ be a sequence of times of impulsive effects such that the number of them during a time interval of length t, denoted by $i(t)$, equals $[\eta(t)]+1$. Here $[\,\cdot\,]$ denotes the integer part of the number, and $\eta(t)$ is a monotone nondecreasing random process such that $\eta(t) \ge \xi(t)$, $\xi(t)$ is a random process that is a solution of a stochastic Ito equation of the form

$$d\xi = a(\xi)dt + \sigma(\xi)dW(t),$$

where the coefficients satisfy the relations

$$a(x) \sim x, \qquad \frac{\sigma}{x} \to 0, \qquad x \to \infty,$$

and

$$\mathbf{P}\{\lim_{t\to\infty} \xi(t) = \infty\} = 1.$$

Conditions that imply that the latter requirements be true are well studied, see e.g. [51, p. 115]. They also imply, see e.g. [51, p. 127], that solutions of the equation satisfy

$$\mathbf{P}\left\{\lim_{t\to\infty}\frac{\ln\xi(t)}{t}=1\right\}=1.$$

Hence, there exists $\Delta(\omega)>0$ such that $\xi(t,\omega)>t^2$ for arbitrary $t>\Delta(\omega)$. Then $i(t,\omega)=[\eta(t)]+1>\xi(t)>t^2$.

Consider the scalar equation

$$\frac{dx}{dt}=f(t,x,\omega)=\begin{cases}-x, & t<\Delta,\\ x, & t\geq\Delta\,.\end{cases}$$

It is clear that its solutions are unstable.

Consider now the impulsive system

$$\begin{aligned}&\frac{dx}{dt}=f(t,x,\omega), && t\neq\tau_i\,,\\ &\Delta x|_{t=\tau_i}=0, && \tau_i<\Delta(\omega),\\ &\Delta x|_{t=\tau_i}=\left(\frac{1}{e}-1\right)x, && \tau_i\geq\Delta(\omega)\,.\end{aligned}$$

It is easy to see that all solutions of this system have the form

$$x(t)=\begin{cases}x_0\exp\{-t\}, & t<\Delta,\\ x_0\exp\{-2\Delta(\omega)+t-i(t)+i(\Delta)\}, & t\geq\Delta,\end{cases}$$

and, since $i(t)>t^2$, we have that $x(t)\to 0$ for $t\to\infty$ with probability 1 and, hence, the impulsive system is totally asymptotically stable with probability 1.

1.4 Stability of systems with permanently acting random perturbations

In a number of works, see e.g. [82], the authors considered the problem of stability of the zero solution when the perturbations occur permanently. More exactly, this means the following.

Let the zero solution of the system

$$\frac{dx}{dt}=F(t,x) \tag{1.34}$$

be stable in a certain sense. Consider the perturbed system

$$\frac{dx}{dt} = F(t,x) + R(t,x). \tag{1.35}$$

The question is whether solutions of this system will lie in a given neighborhood of the origin for $t > t_0$ if $R(t,x)$ is sufficiently small. A similar problem for deterministic systems with impulsive effects was considered in [143]. For random perturbations, this problem was considered in [70].

Consider this problem for a system with random impulsive perturbations.

Let a deterministic system with random impulsive perturbations,

$$\begin{aligned} \frac{dx}{dt} &= F(t,x), \qquad t \neq t_i, \\ \Delta x|_{t=t_i} &= I_i(x), \end{aligned} \tag{1.36}$$

have a trivial solution. We will assume that the sequence of times of the impulses does not have finite limit points. Together with (1.36), consider the perturbed system

$$\begin{aligned} \frac{dx}{dt} &= F(t,x) + R(t,x,\omega), \qquad t \neq t_i, \\ \Delta x|_{t=t_i} &= I_i(x) + J_i(x,\omega), \end{aligned} \tag{1.37}$$

where $R(t,x,\omega)$ is a random process for any fixed $x \in \mathbf{R}^n$, and $J_i(x,\omega)$ is a random variable.

Let the random process

$$\zeta(t) = \sup_{x \in \mathbf{R}^n} |R(t,x,\omega)|$$

and the sequence of random variables

$$\eta_i = \sup_{x \in \mathbf{R}^n} |J_i(x,\omega)|$$

have finite expectations.

Definition 1.4. A solution $x(t) \equiv 0$ of system (1.36) is called *stable under an action of permanently acting perturbations* if for arbitrary $\varepsilon > 0$ and $\Delta > 0$ there exists $\gamma > 0$ such that, if

$$|x_0| + \sup_{t \geq t_0} \mathbf{E}\zeta(t,\omega) + \sup_{i \in \mathbb{N}} \mathbf{E}\eta_i(\omega) < \gamma$$

for $t > t_0$, then

$$\mathbf{P}\left\{|x(t,t_0,x_0)| > \Delta\right\} < \varepsilon.$$

Again, a stability condition is formulated in terms of a Lyapunov function. We will assume that system (1.37) satisfies a condition for existence and uniqueness of a solution for $x \in \mathbf{R}^n$, $t \geq t_0$.

Theorem 1.7. *Let there exist a function $V(t,x)$ on a domain $x \in \mathbf{R}^n$, $t \geq t_0$ for some $t_0 \in \mathbf{R}$, which is absolutely continuous in t and globally Lipschitz continuous in x with a Lipschitz constant L, as well as the following conditions:*

1)

$$V(t,0) \equiv 0, \quad \inf_{|x|>\delta, t>t_0} V(t,x) = V_\delta > 0, \ \ \delta > 0;$$

2) for an arbitrary $\delta > 0$ there exists $C_\delta > 0$ such that

$$\frac{d^0 V}{dt} \leq \ -C_\delta V(t,x)$$

on the domain $(|x| > \delta) \times (t > t_0)$;

3)

$$V(t_i, x + I_i(x)) - V(t_i, x) \leq -CV(t_i, x)$$

for $x \in \mathbf{R}^n$, $C > 0$.

Then the solution $x \equiv 0$ of system (1.36) *is stable under permanently acting perturbations.*

Proof. We will assume that $t_0 = 0$. Denote

$$V^\delta = \sup_{t>0, |x|<\delta} V(t,x).$$

Then $V^\delta \to 0$ for $\delta \to 0$.

Let $x(t,x_0)$ be a solution of system (1.37) such that $x(0,x_0) = x_0$. If t_1 is the time moment of the first impulse, then

$$\mathbf{E}V(t, x(t,x_0)) \leq V(0,x_0)\exp\{-C_\delta t\} + \frac{C}{C_\delta}\sup_{t\geq 0}\mathbf{E}\zeta(t) + V^\delta \tag{1.38}$$

on the interval $[0,\, t_1)$, as in [70]. For $t = t_1$, we have

$$V(t_1+0, x(t_1+0) \leq V(t_1, x(t_1))(1-C)\eta_1 + L\eta_1. \tag{1.39}$$

On the following intervals $(t_i, t_{i+1}]$, using (1.38) and (1.39) we have

$$\begin{aligned} \mathbf{E}V(t, x(t)) \leq (L \sup_{i\in\mathbf{N}} E\eta_i + V(0, x_0 + \frac{C}{C_\delta}\sup_{t\geq 0})\mathbf{E}\zeta(t) \\ + V^\delta(1 + (1-C) + (1-C)^2 + \cdots + (1-C)^i)\,. \end{aligned} \tag{1.40}$$

The rest of the proof is similar to the corresponding fact in [70, p. 50]. □

1.5 Solutions periodic in the restricted sense

In this section we will consider conditions that imply that system (1.1) has solutions periodic in the restricted sense. These conditions were obtained in [122].

Consider system (1.1) satisfying conditions of Theorem 1.1.

Let also the following conditions be satisfied:

1) the function f is periodic in t with period T;

2) for some natural p, the following holds:

$$I_{i+p}(x,z) = I_i(x,z),\ t_{i+p} = t_i + T; \tag{1.41}$$

3) $\xi(t)$ and η_i are periodically connected, that is,

$$\begin{aligned}&\mathbf{P}\{\xi(s_1+T)\in A_1,\dots\xi(s_m+T)\in A_m, \eta_{l_1+p}\in B_1,\dots\eta_{l_r+p}\in B_r\}\\ &\qquad = \mathbf{P}\{\xi(s_1)\in A_1,\dots\xi(s_m)\in A_m, \eta_{l_1}\in B_1,\dots\eta_{l_r}\in B_r\}.\end{aligned}$$

for arbitrary $s_1,\dots s_m \in \mathbf{R}$, $l_1,\dots l_r \in Z$, $A_1,\dots A_m \in B(\mathbf{R}^k)$, $B_1,\dots B_r \in B(\mathbf{R}^l)$, where $B(\mathbf{R}^k)$ and $B(\mathbf{R}^l)$ are Borel σ-algebras on $\mathbf{R}^k$ and $\mathbf{R}^l$, respectively.

Conditions 1)–3) will be called periodicity conditions. For T-periodic systems of type (1.1), we have the following theorem.

Theorem 1.8. *Let system* (1.1) *satisfy conditions of Theorem 1.1. Then it has a solution that is T-periodic and periodically connected with $\xi(t)$ and η_i if and only if this system has a solution $y(t)$ satisfying*

$$\frac{1}{|k|+1}\sum_{i=1}^{k}\mathbf{P}\{|y(iT)|>r\}\to 0 \tag{1.42}$$

as $r\to\infty$ uniformly in $k = 1,2,\dots$ or $k=-1,-2,\dots$.

Proof. It is clear that condition (1.42) is necessary, since it is satisfied for a T-periodic solution of system (1.1).

To prove sufficiency, we will use ideas of [70], where a similar result is obtained for systems without impulsive effects.

Take an arbitrary $t>0$ and consider a solution of system (1.1) on the interval $[0,\ t]$ with the initial condition $x(0)=x_0(\omega)$. Solving system (1.1) on the interval $[0,t_1]$, where t_1 is the first impulse time in $[0,t]$, using successive approximations we can see that the random variable $x(s)$ is measurable with respect to the minimal σ-algebra containing the events $\{\xi(u) \in A_1\}(u \in [0\ s], s \le t_1)$ and $\{x_0(\omega) \in A_2\}$, where A_1 is a Borel subset of $\mathbf{R}^k$ and A_2 is a Borel subset of $\mathbf{R}^n$. By considering a solution of system (1.1) on the interval $[t_1, t_2]$ satisfying the initial conditions $x(t_1+) = x(t_1) + I_1(x(t_1), \eta_1)$, in a similar way we see that it is measurable with respect to a minimal σ-algebra that contains the events $\{\xi(u) \in A_1\}(u \in [0\ s], s \le t_2)$ and $\{x_0(\omega) \in A_2\}, \{\eta_1 \in A_3\}$, where A_3 is a Borel subset of $\mathbf{R}^l$. Continuing this process we verify that the random variable $x(t)$ is measurable with respect to a minimal σ-algebra that contains the events $\{\xi(s) \in A_1\}(s \in [0\ t]$ and $\{x_0(\omega) \in A_2\}, \{\eta_i \in A_3\}$, where t_i are times of the impulsive effects in $[0, t]$.

Using conditions of the theorem and the representation of $x(t+T)$ in terms of a sum and an integral we have that

$$\begin{aligned}
x(t+T) &= x(T) + \int_T^{T+t} f(s, x(s), \xi(s))ds + \sum_{T<t_i<t+T} I_i(x(t_i), \eta_i) \\
&= x(T) + \int_0^t f(s, x(s+T), \xi(s+T))ds + \sum_{0<t_i<t} I_{i+p}(x(t_i+T), \eta_{i+p}) \\
&= x(T) + \int_0^t f(s, x(s+T), \xi(s+T))ds + \sum_{0<t_i<t} I_i(x(t_i+T), \eta_{i+p})
\end{aligned}$$

and

$$x(t) = x(0) + \int_0^t f(s, x(s), \xi(s))ds + \sum_{0<t_i<t} I_i(x(t_i), \eta_i)\,.$$

So to prove the theorem it is sufficient to show that there is a random variable $\zeta(\omega)$ such that

$$\begin{aligned}
&\mathbf{P}\{\zeta(\omega) \in A_0, \xi(s_1) \in A_1, \ldots, \xi(s_m) \in A_m, \eta_{l_1} \in B_1, \ldots, \eta_{l_q} \in B_q\} \\
&\quad = \mathbf{P}\{x(T) \in A_0, \xi(s_1+T) \in A_1, \ldots, \xi(s_m+T) \in A_m, \eta_{l_1+p} \\
&\qquad \in B_1, \ldots, \eta_{l_q+p} \in B_q\} \qquad (1.43)
\end{aligned}$$

for arbitrary

$$t > 0, A_0, A_1, \ldots A_m, B_1, B_2, \ldots B_q, s_1, \ldots s_m, l_1, \ldots l_q\,.$$

Here $x(t)$ is a solution of system (1.1) satisfying $x(0) = \zeta(\omega)$.

Let, for example, condition (1.42) hold for $k > 0$. Denote by $\eta(t)$ a random process that coincides with η_i on the intervals $[t_i,\ t_{i+1})$. Clearly, it is stochastically right continuous and periodically connected with $\xi(t)$. Introduce a random variable τ_k, which is independent of $\xi(t)$, η_i, and $y(0)$ such that

$$\mathbf{P}\{\tau_k = nT\} = \frac{1}{k+1}\ (n = \ 0, 1, \dots, k)$$

and

$$x_0^k = y(\tau_k), x_k(t) = y(t+\tau_k), \xi_k(t) = \xi(t+\tau_k)\,, \eta_k(t) = \eta(t+\tau_k)\,.$$

Using the formula for complete probability, similarly to [70] one can show that the family $(x_0^{(k)}, \xi_k(t), \eta_k(t))$ satisfies the conditions of the theorem in [155, p. 13] on weak compactness and convergence in probability. Hence, there exists a sequence $(\overline{x}_0^{(k)}, \overline{\xi}_k(t), \overline{\eta}_k(t))$ in some probability space $(\overline{\Omega}, \overline{F}, \overline{\mathbf{P}})$ having the same distribution as $(x_0^{(k)}, \xi_k(t), \eta_k(t))$ and containing a subsequence $(\overline{x}_0^{(n_k)}, \overline{\xi}_{n_k}(t), \overline{\eta}_{n_k}(t))$ that converges in probability to $(\overline{x}, \overline{\xi}(t), \overline{\eta}(t))$. Similarly to [70], one can show that the finite dimensional distributions of $(\overline{\xi}(t), \overline{\eta}(t))$ coincide with the distributions of $(\xi(t), \eta(t))$. Let us now construct random variables $x(\omega), x^{(n_k)}$ on the initial probability space such that their joint distribution with $\xi(t), \eta(t)$ be the same as the joint distribution of $(\overline{x}, \overline{x}_0^{(n_k)}\overline{\xi}(t), \overline{\eta}(t))$.

Let us now show that condition (1.43) is satisfied for $\zeta(\omega) = x(\omega)$. Denote by $x_{n_k}(t)$ a solution of system (1.1) such that $x_{n_k}(0) = x^{(n_k)}(\omega)$ and prove that $x_{n_k}(t)$ converges to $x(t)$ in probability for every fixed t.

Indeed, since $x_{n_k}(t)$ and $x(t)$ are solutions of system (1.1), they can be written on the interval $[0,\ t]$ as [143, p. 20]

$$x(t) = x(\omega) + \int_0^t f(s, x(s), \xi(s))ds + \sum_{0<t_i<t} I_i(x(t_i), \eta_i)\,, \tag{1.44}$$

$$x_{n_k}(t) = x^{n_k}(\omega) + \int_0^t f(s, x_{n_k}(s), \xi(s))ds + \sum_{0<t_i<t} I_i(x_{n_k}(t_i), \eta_i)\,. \tag{1.45}$$

Subtracting (1.44) from (1.45) and using the Lipschitz condition we get that

$$|x_{n_k}(t) - x(t)| \le |x^{n_k} - x| + \int_0^t B(s)|x_{n_k}(s) - x(s)|ds + \sum_{0<t_i<t} L_i(\omega)|x_{n_k}(t_i) - x(t_i)|\,.$$

Using a generalization of the Gronwall–Bellman inequality [143, p. 16] we obtain that

$$|x_{n_k}(t) - x(t)| \le |x^{n_k} - x| \prod_{0<t_i<t} (1 + L_i(\omega)) \exp\left\{ \int_0^t B(s,\omega)ds \right\}. \qquad (1.46)$$

Since, by conditions of the theorem, the number of impulses on the interval $[0, t]$ is countable, $B(s, \omega)$ is locally integrable with probability 1, we see that the two last factors in formula (1.46) are bounded in probability on $[0, t]$. So, $x_{n_k}(t) \to x(t)$ as $k \to \infty$ for each fixed t. The rest of the theorem is proved as in [70]. □

Remark 1. Conditions of Theorem 1.8 are clearly satisfied if system (1.1) is dissipative.

Remark 2. Let the functions f and I_i in system (1.1) be deterministic, so that (1.1) is a deterministic T-periodic impulsive system of the form

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), \qquad t \ne t_i\,, \\ \Delta x|_{t=t_i} &= I_i(x)\,. \end{aligned} \qquad (1.47)$$

If this system has at least one bounded solution then, as follows from Theorem 1.8, it also has a solution that is T-periodic in probability sense satisfying, generally speaking, a random initial condition. This, of course, does not necessarily imply existence of a periodic deterministic solution, since it may happen that a periodic random process does not have periodic trajectories.

Remark 3. Solutions of a system of ordinary differential equations and solutions of an impulsive system can be qualitatively very different because of the impulsive effects. Let us give relative examples.

Example 1. Consider that scalar equation

$$\frac{dx}{dt} = 1\,.$$

All of its solutions have the form $x(t) = t + x_0(\omega)$ and monotonically approach infinity with probability 1 as t increases. Hence, they are not bounded in probability. So, this equation does not have solutions periodic in the above sense.

However, the impulsive system

$$\begin{aligned} \frac{dx}{dt} &= 1, \qquad t \ne i\,, \\ \Delta x|_{t=i} &= -1 \end{aligned}$$

where i is an integer, has the periodic solution $x = \{t\}$.

Example 2. Consider the scalar equation

$$\frac{dx}{dt} = 0\,.$$

All its solutions are constants, hence, periodic with an arbitrary period. At the same time, the impulsive system

$$\frac{dx}{dt} = 0, \qquad t \neq i\,,$$
$$\Delta x|_{t=i} = 1$$

has no periodic solution, since all its solutions are unboundedly increasing with probability 1.

Although the conditions of Theorem 1.8 are necessary and sufficient, they are not very effective. However, in a particular case where system (1.1) is of the form

$$\begin{aligned} \frac{dx}{dt} &= f(t,x) + \sigma(t,x)\xi(t), \qquad t \neq t_i\,, \\ \Delta x|_{t=t_i} &= I_i(x) + J_i(x)\eta_i\,, \end{aligned} \tag{1.48}$$

one can formulate effective conditions for existence of periodic solutions in terms of Lyapunov functions for the truncated deterministic system

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), \qquad t \neq t_i\,, \\ \Delta x|_{t=t_i} &= I_i(x)\,. \end{aligned} \tag{1.49}$$

Theorem 1.9. *Let there exist a nonnegative function $V(t,x)$ on the domain $x \in \mathbf{R}^n$, $t \geq t_0$, for some $t_0 \in \mathbf{R}$. Let it be absolutely continuous in t, globally Lipschitz continuous in x, as well as satisfy the conditions of Theorem 1.1. Let also the following be satisfied:*

a)

$$V_r = \inf_{|x|>r, t>t_0} V(t,x) \to \infty,\ r \to \infty\,;$$

b)

$$\frac{d^0 V}{dt} \leq\ -C_1 V,\ V(t_i, x + I_i(x)) - V(t_i,x) \leq -C_2 V(t_i,x)\,;$$

c)

$$||\sigma(t,x)|| +\ ||J_i(x)|| \leq C_3\,,$$

where C_1, C_2, C_3 are positive constants. Let f and σ be T-periodic in t.

Then system (1.48) *has a solution that is T-periodic in the restricted sense for an arbitrary stochastically continuous process $\xi(t)$ and a sequence η_i such that they are T-periodically connected, and $\mathbf{E}|\xi(t)| + \mathbf{E}|\eta_i| < \infty$.*

A proof of this theorem follows from Theorem 1.2 and Remark 1.

1.6 Periodic solutions of systems with small perturbations

In this section, we will use Theorem 1.8 to study existence of periodic solution in some classes of systems with random impulsive effects, more exactly, systems with small random perturbations.

Let us find a relation between an asymptotically stable, compact invariant set of a deterministic autonomous system and periodic solutions of a perturbed system obtained from the deterministic system by introducing small continuous and impulsive random perturbations.

Consider the autonomous system

$$\frac{dx}{dt} = F(x) \tag{1.50}$$

defined on a domain $D \subset \mathbf{R}^n$. Assume that the function F is Lipschitz continuous on this domain, and system (1.50) has an asymptotically stable, compact invariant set S.

Consider the perturbed periodic impulsive system

$$\begin{aligned} \frac{dx}{dt} &= F(x) + \varepsilon g(t, x, \xi(t)), \qquad t \neq t_i, \\ \Delta x|_{t=t_i} &= \varepsilon I_i(x, \eta_i), \end{aligned} \tag{1.51}$$

where $\varepsilon > 0$ is a small parameter, the functions $g(t,x,y)$ and $I_i(x,z)$ are bounded for $t \in \mathbf{R}$, $x \in D$, $y \in \mathbf{R}^k$, $z \in \mathbf{R}^l$, and, together with the random process $\xi(t)$, the sequence of random variables η_i, and the times of the impulsive effects, satisfy the periodicity conditions 1)–3) in the previous section. Let also system (1.51) satisfy a condition that implies existence and trajectory-wise uniqueness of a solution of a Cauchy problem on the domain D.

Theorem 1.10. *Let system* (1.51) *satisfy the above conditions.*

Then, for small values of the parameter ε, the system has a solution $x(t)$, which is T-periodic and periodically connected with $\xi(t)$ and η_i, such that

$$\sup_{-\infty < t < \infty} \rho(x(t), S) < \delta(\varepsilon), \qquad \delta(\varepsilon) \to 0, \quad \varepsilon \to 0\,, \tag{1.52}$$

with probability 1.

Theorem 1.10 thus states that an asymptotically stable invariant set gives rise to a solution of a perturbed system, periodic in the restricted sense. For systems without impulsive effects, a close result was obtained in [70].

Proof of Theorem 1.10. Represent a solution of system (1.51) as

$$x(t, x_0) = x_0 + \int_0^t F(x(s)) + \varepsilon g(t, x(s), \xi(s))\, ds + \varepsilon \sum_{0<t_i<t} I_i(x(t_i), \eta_i)\,. \quad (1.53)$$

Denote by $y(t, x_0)$ a solution of system (1.50). Since S is an asymptotically stable manifold for system (1.50), for an arbitrary $\mu > 0$ there exist numbers $\nu > 0$ and $A > 0$ such that

$$\rho(y(t, x_0), S) < \frac{\mu}{2}, \qquad t \geq 0\,, \quad (1.54)$$

and

$$\rho(y(t, x_0), S) < \frac{\nu}{2}, \qquad t \geq A\,, \quad (1.55)$$

if $\rho(x_0, S) \leq \nu$.

Let $\tau_D(x_0)$ be the time at which the solution $x(t, x_0)$ hits the boundary of the domain D.

Let $A_1 = mT$ be a number that is the closest to A, a multiple of the period T and such that $A_1 \geq A$. Let us estimate the difference between solutions of the unperturbed and the perturbed systems, $y(t, x_0)$ and $x(t, x_0)$, on the segment $[0, A_1]$. Since $g(t, x, y)$ and $I_i(x, z)$ are bounded with some constant $K > 0$, we have

$$|x(t, x_0) - y(t, x_0)| \leq \int_0^t |F(x(s, x_0)) - F(y(s, x_0))|\, ds + \varepsilon(KA_1 + Kmp)$$

with probability 1 for $t \leq \tau_D(x_0)$. Using the Gronwall-Bellman inequality we get the estimate

$$|x(t, x_0) - y(t, x_0)| \leq \varepsilon(KA_1 + Kmp) \exp\{LA_1\}\,, \quad (1.56)$$

which holds for all $t \leq \tau_D(x_0)$.

Choose $\mu > 0$ small enough so that the μ-neighborhood of the set S is contained in the domain D. Using this μ, choose ν, $0 < \nu < \mu$, such that the inequalities (1.54) and (1.55) would hold. Finally, take $\varepsilon > 0$ such that

$$\varepsilon(KA_1 + Kmp) \exp\{LA_1\} < \frac{\nu}{2}. \quad (1.57)$$

Then

$$\rho(x(t,x_0),S) \leq |x(t,x_0) - y(t,x_0)| + \rho(y(t,x_0),S) \leq \frac{\nu}{2} + \frac{\mu}{2} < \mu, \qquad (1.58)$$

for x_0 belonging to the ν-neighborhood of the set S.

Let us show that $\tau_D(x_0) > A_1$ with probability 1 for x_0 in the ν-neighborhood of the set S. Assume the contrary. Then there exists a set B such that $\mathbf{P}(B) > 0$ and, for all $\omega \in B$, the solution $x(t,x_0,\omega)$ hits the boundary of the domain D in time less than A_1. Then there exists a time $t_0(\omega)$ such that

$$\rho(x(t_0(\omega),x_0,\omega),S) = \mu. \qquad (1.59)$$

Since $x(t_0(\omega),x_0,\omega) \in D$ at the time $t_0(\omega)$, it follows from inequality (1.58) that

$$\mu = \rho(x(t_0(\omega),x_0,\omega),S) < \mu$$

for $t = t_0(\omega)$. This contradiction proves that solutions of (1.51) that start in a ν-neighborhood of the set S do not leave the domain D until the time A_1 with probability 1. Hence, inequality (1.58) holds for an arbitrary $t \in [0, A_1]$, and for $t = A_1$, we get from (1.55) and (1.58) that

$$\rho(x(A_1,x_0),S) < \nu. \qquad (1.60)$$

Hence, the solution $x(t,x_0)$ of system (1.51), having left the ν-neighborhood of the set S, does not leave its μ-neighborhood with probability 1 for $t \in [0,\ A_1]$, and again enters the ν-neighborhood of this set at $t = A_1$.

Consider now the behavior of the solution $x(t,x_0)$ on the line segment $[A_1, 2A_1]$. To this end, we will compare it with a solution $y_1(t)$ of system (1.50) such that $y_1(A_1) = x(A_1,x_0)$.

By using, as in [198], that the limit in the definition of asymptotic stability of a compact manifold of system (1.50) is uniform in x_0 that belongs to a ν-neighborhood of the set S and that the number of impulses on an arbitrary time interval of length mT is the same, making the same reasoning as before applied to the difference $x(t,x_0) - y_1(t)$ we get estimate (1.56) on the line segment $[A_1, 2A_1]$ and then (1.58) and (1.60). Hence, the solution $x(t,x_0)$, which is in an ν-neighborhood of the set S with probability 1 at $t = A_1$, does not leave the μ-neighborhood of this set with probability 1 for $t \in [A_1, 2A_1]$ and, again, entering the ν-neighborhood of the set S at the time $t = 2A_1$.

Continuing this process we obtain that the solution $x(t,x_0)$, which leaves the ν-neighborhood of the set S, does leave the μ-neighborhood of this set. The latter means that system (1.51) has a solution bounded for $t \geq 0$ that does not leave the μ-neighborhood of the set S. This shows that $x(t, x_0)$ satisfies condition (1.42), which is sufficient for existence of a T-periodic solution. It follows from the construction of such a solution that it belongs the μ-neighborhood of the set S. By choosing $\mu = \mu(\varepsilon)$ to be monotone in ε, we obtain estimate (1.52). □

This theorem yields existence of a periodic solution of a perturbed system if these perturbations are small with probability 1. Let us now consider the case where both the perturbations in the right-hand side and the impulsive effects are small in the mean. In this case, for the invariant set used in the proof of Theorem 1.10 to be stable under permanently acting perturbations, as follows from [70, p. 48], it is not sufficient already that the set be asymptotically stable. One needs to impose a stricter condition — require it to be exponentially stable.

So, let us again consider system (1.51) satisfying the conditions of Theorem 1.10 without the condition that $g(t, x, y)$ and $I_i(x, z)$. Denote by

$$a(t, \omega) = \sup_{x \in \mathbf{R}^n} |g(t, x, \xi(t))|, \ \ b_i(\omega) = \sup_{x \in \mathbf{R}^n} |I_i(x, \eta_i(\omega))|.$$

Also assume that the random process $a(t)$, the sequence of random variables b_i have finite expectations, and there exists $C > 0$ such that

$$\sup_{t \in [0,\ T]} \mathbf{E}a(t) + \sup_{i = \overline{1,p}} \mathbf{E}b_i \leq C\,.$$

We will also assume that the functions F, g, I_i are defined and Lipschitz continuous with respect to x on $\mathbf{R}^n$.

Theorem 1.11. *Let the invariant set of system* (1.50) *be totally exponentially stable.*

Then, for small values of the parameter ε, system (1.51) *has a T-periodic solution $x(t)$, periodically connected with $\xi(t)$ and η_i such that*

$$\sup_{t \in [0,\ T]} \mathbf{E}\rho(x(t), S) < \delta(\varepsilon), \qquad \delta(\varepsilon) \to 0, \quad \varepsilon \to 0\,. \tag{1.61}$$

Proof. Since the invariant set S of system (1.50) is totally exponentially stable, an arbitrary solution $y(t, x_0)$ of this system satisfies the estimate

$$\rho(y(t, x_0), S) \leq K \exp\{-\gamma(t - \tau)\}\rho(y(\tau, x_0), S) \tag{1.62}$$

for any $t \geq \tau \geq 0$, where the constants γ and K are positive are independent of t, τ, and x_0.

Let us choose a number $A > 0$, which is a multiple of the period T, $A = mT$, so large that

$$K \exp\{-\gamma A\} < \frac{1}{2}. \tag{1.63}$$

Let $x(t, x_0)$ be a solution of system (1.51). Let us find an estimate for the mean of the distance between this solution and the set S over $[0, A]$. As in Theorem 1.10, we have

$$\begin{aligned}
\mathbf{E}|x(t,x_0) - y(t,x_0)| &\leq \int_0^t L\mathbf{E}|x(s,x_0) - y(s,x_0)|\, ds \\
&\quad + \varepsilon \int_0^A \mathbf{E}|g(s, x(s,x_0), \xi(s))|\, ds + \varepsilon \sum_{0<t_i<A} \mathbf{E}|I_i(x(t_i,x_0), \eta_i)| \\
&\quad \leq L \int_0^t \mathbf{E}|x(s,x_0) - y(s,x_0)|\, ds + \varepsilon \int_0^A \mathbf{E}a(t)\, dt + \sum_{0<t_i<A} \mathbf{E}b_i \\
&\quad \leq L \int_0^t \mathbf{E}|x(s,x_0) - y(s,x_0)|\, ds + \varepsilon(AC + mpC)\,.
\end{aligned}$$

This gives the estimate

$$\mathbf{E}|x(t,x_0) - y(t,x_0)| \leq \varepsilon(AC + mpC)\exp\{LA\}, \tag{1.64}$$

which holds for $t \in [0,\ A]$.

Hence, for an arbitrary $t \in [0, A]$, we have

$$\begin{aligned}
\mathbf{E}\rho(x(t,x_0), S) &\leq \rho(y(t,x_0), S) + \varepsilon(AC + mpC)\exp\{LA\} \\
&\leq K\exp\{-\gamma t\}\rho(x_0, S) + \varepsilon(AC + mpC)\exp\{LA\}\,.
\end{aligned} \tag{1.65}$$

By requiring that $\rho(x_0, S) < \nu(\varepsilon)$, where $\nu(\varepsilon)$ is sufficiently small and monotone with respect to ε, we finally obtain that

$$\mathbf{E}\rho(x(t,x_0), S) < \delta(\varepsilon) \tag{1.66}$$

for some function $\delta \to 0$, $\varepsilon \to 0$ and $t \in [0, A]$.

To estimate the behavior of the solution $x(t, x_0)$ on $[A, 2A]$, consider a solution $y_1(t)$ of system (1.50) such that

$$y_1(A) = x(A, x_0).$$

As before, using (1.62) we get a chain of inequalities,

$$\begin{aligned}\mathbf{E}\rho(x(t,x_0),S) &\le \rho(y_1(t),S) + \varepsilon(AC+mpC)\exp\{LA\}\\ &\le K\exp\{-\gamma(t-A)\}\mathbf{E}\rho(y_1(A),S) + \varepsilon(AC+mpC)\exp\{LA\}\\ &\le K\exp\{-\gamma(t-A)\}\delta(\varepsilon) + \varepsilon(AC+mpC)\exp\{LA\} \le (K+1)\delta(\varepsilon).\end{aligned}$$

For $t = 2A$, using (1.63) we have

$$\mathbf{E}\rho(x(t,x_0),S) \le \left(\frac{1}{2}+1\right)\delta(\varepsilon)\,.$$

Consider a solution $y_2(t)$ of system (1.50) on the segment$[2A,\ 3A]$ satisfying $y_2(2A) = x(2A,x_0)$. Then, for $t \in [2A,\ 3A]$, we have

$$\begin{aligned}\mathbf{E}\rho(x(t,x_0),S) &\le K\exp\{-\gamma(t-2A)\}\mathbf{E}\rho(y_2(2A),S) + \varepsilon(AC+mpC)\exp\{LA\}\\ &\le K\left(\frac{1}{2}+1\right)\delta(\varepsilon) + \delta(\varepsilon) = \left(K\left(\frac{1}{2}+1\right)+1\right)\delta(\varepsilon).\end{aligned}$$

For $t = 3A$, we similarly get

$$\mathbf{E}\rho(x(3A,x_0),S) \le \frac{1}{2}\left(\frac{1}{2}+1\right)\delta(\varepsilon) + \delta(\varepsilon) = \left(1+\frac{1}{2}+\frac{1}{4}\right)\delta(\varepsilon)\,.$$

Continuing this process to the segment $[(k-1)A, kA]$ we obtain the estimates

$$\mathbf{E}\rho(x(t,x_0),S) \le \left(K\left(1+\frac{1}{2}+\cdots+\left(\frac{1}{2}\right)^{k-2}\right)+1\right)\delta(\varepsilon) \tag{1.67}$$

and

$$\mathbf{E}\rho(x(kA,x_0),S) \le \left(1+\frac{1}{2}+\cdots+\left(\frac{1}{2}\right)^{k-1}\right)\delta(\varepsilon) \tag{1.68}$$

for $t = kA$.

It follows from (1.67) and (1.68) that there exists a positive constant C_1 such that

$$\mathbf{E}\rho(x(t,x_0),S) \le C_1\delta(\varepsilon) \tag{1.69}$$

for arbitrary $t \ge 0$. Using (1.69) and Chebyshev's inequality we now see that condition (1.42) holds for $x(t,x_0)$, which implies that system (1.51) has a periodic solution $x(t)$ that is periodically connected with $\xi(t)$ and η_i.

Constructing the initial value of this solution as in Theorem 1.8 it is easy to see that $M\rho(x(0),S)$ can be made arbitrarily small by requiring ε and $\nu(\varepsilon)$ to be small, where $x(0)$ is the initial value of the periodic solution. By making estimates for the solution on $[0,T]$ similarly to (1.64) and (1.65), we see that $M\rho(x(t),S)$ is finite, and an estimate of type (1.66) holds. $\square$

1.7 Periodic solutions of linear impulsive systems

Necessary and sufficient conditions for existence of solutions periodic in the restricted sense of linear differential systems or systems perturbed with random processes were considered in [41]. Similar problems for impulsive systems were considered in [142], the results of which we now will be discussing.

Let system (1.1) satisfy the periodicity conditions 1)–3) and has the form

$$\begin{gathered}\frac{dx}{dt} = A(t)x + \xi(t), \qquad t \neq t_i\,, \\ \Delta x|_{t=t_i} = B_i x + \eta_i\,,\end{gathered} \tag{1.70}$$

where the random process $\xi(t)$ and the sequence of random variables η_i are such that

$$\int_0^T \mathbf{E}|\xi(t)|\,dt < \infty, \ \sum_{i=1}^{p} \mathbf{E}|\eta_i| < \infty\,. \tag{1.71}$$

We consider $A(t)$, B_i, $\xi(t)$, η_i to be complex-valued in general.

Denote by $X(t)$ the matriciant of the linear homogeneous system

$$\begin{gathered}\frac{dx}{dt} = A(t)x, \qquad t \neq t_i\,, \\ \Delta x|_{t=t_i} = B_i x\,.\end{gathered} \tag{1.72}$$

Theorem 1.12. *For system* (1.70) *with the random process* $\xi(t)$ *and the sequence of random variables* η_i *considered above to have a solution* $x(t)$ *unique, up to stochastic equivalence,* T*-periodic and periodically connected with* $\xi(t)$ *and* η_i *and such that*

$$\sup_{0\le t\le T} \mathbf{E}|x(t)| < \infty,$$

it is necessary and sufficient that the spectrum of the monodromy matrix $X(T)$ *of system* (1.72) *would not intersect the unit circle* $S = \{\lambda \in C|\ |\lambda| = 1\}$*, that is,*

$$\sigma(X(T)) \cap S = \emptyset. \tag{1.73}$$

Proof. As follows from [143, p. 87] there is an analogue of the Floquet-Lyapunov theorem for systems of type (1.72). By using it, we can apply a nondegenerate, piecewise smooth, and periodic Lyapunov change of variables $x = \Phi(t)y$ to reduce system (1.72) to a system with constant coefficients without impulsive effects,

$$\frac{dy}{dt} = Py, \tag{1.74}$$

and such that

$$\frac{1}{T}\ln|\rho_i| = Re\lambda_i(P),$$

where ρ_i is a multiplier of system (1.72). Using this representation and conditions of the theorem we see that the spectrum of the matrix P does not intersect the imaginary axis.

Let us make a change of variables,

$$x = \Phi(t)y\,, \tag{1.75}$$

in system (1.70). Then we get

$$\frac{dy}{dt} = Py + \Phi^{-1}(t)\xi(t)$$

for $t \neq t_i$. For $t = t_i$, using that $\Phi(t) = X(t)\exp\{-Pt\}$, we get

$$\begin{aligned} x(t_i+0) - x(t_i) &= X(t_i+0)\exp\{-Pt_i\}y(t_i+0) - X(t_i)\exp\{-Pt_i\}y(t_i) \\ &= B_iX(t_i)\exp\{-Pt_i\}y(t_i) + \eta_i\,. \end{aligned}$$

Whence,

$$X(t_i+0)\exp\{-Pt_i\}y(t_i+0) = (B_i+E)X(t_i)\exp\{-Pt_i\}y(t_i) + \eta_i\,.$$

But, since

$$X(t_i+0) - X(t_i) = B_iX(t_i)$$

by (1.72), we have that

$$X(t_i+0)\exp\{-Pt_i\}\Delta y|_{t=t_i} = \eta_i$$

or

$$\Delta y|_{t=t_i} = \Phi^{-1}(t_i+0)\eta_i\,.$$

Hence, the change of variables (1.75) reduces system (1.70) to the form

$$\begin{aligned} \frac{dy}{dt} &= Py + \Phi^{-1}(t)\xi(t), \qquad t \neq t_i\,, \\ \Delta y|_{t=t_i} &= \Phi^{-1}(t_i+0)\eta_i\,. \end{aligned} \tag{1.76}$$

But since $\Phi(t)$ is piecewise continuous and periodic with period T, the change of variables (1.75) takes periodic processes into periodic, and the process $\Phi^{-1}(t)\xi(t)$ is T-periodic and periodically connected with the periodic sequence $\Phi^{-1}(t_i+0)\eta_i$. Denote

$$\Phi^{-1}(t)\xi(t) = \theta(t)$$

and

$$\Phi^{-1}(t_i + 0)\eta_i = \zeta_i.$$

Then system (1.76) can be written as

$$\begin{gathered} \frac{dy}{dt} = Py + \theta(t), \qquad t \neq t_i\,, \\ \Delta y|_{t=t_i} = \zeta_i\,, \end{gathered} \tag{1.77}$$

where the spectrum of the matrix P does not intersect the imaginary axis.

Hence, a study periodic solutions of system (1.70) becomes equivalent to a study of such solutions for system (1.77).

Let us prove necessity of the conditions of the theorem.

Necessity. By the conditions of the theorem, equation (1.77) has a unique periodic solution for arbitrary $\theta(t)$ and ζ_i that satisfy the conditions of the theorem. Set $\zeta_i = 0$ and let $\theta(t)$ to be the random process

$$-\exp\{i(\alpha t + \tau)\}y_0\,, \tag{1.78}$$

where $\alpha \in \mathbf{R}$, $y_0 \in \mathbf{C}^n$, and τ is a random variable uniformly distributed on $[0, 2\pi]$. As follows from [70, p. 70], this processes will be stationary and, hence, periodic with an arbitrary period.

Let $y(t)$ be a T-periodic solution of system (1.77) with process (1.78). It follows from the conditions of the theorem that the process

$$\{y(t), \exp\{i(\alpha t + \tau)\}\}$$

will also be periodic.

Consider now the random variable τ_1 that is independent of the process $\{y(t), \exp\{i(\alpha t + \tau)\}\}$ and uniformly distributed on $[0, T]$. Then it follows from [70, p. 70] that the process

$$\{y(t + \tau_1), \exp\{i(\alpha(t + \tau_1) + \tau)\}\}$$

is stationary. Denote $z(t) = y(t + \tau_1)$. It is clear that $z(t)$ satisfies the system

$$\frac{dz}{dt} = Pz - \exp\{i(\alpha(t + \tau_1) + \tau)\}\}y_0\,.$$

Whence we have

$$z(t)\exp\{-i\tau - i\alpha(t+\tau_1)\} - z(0)\exp\{-i\tau - i\alpha\tau_1\}$$
$$= \int_0^t (P - i\alpha)z(s)\exp\{-i\tau - i\alpha(s+\tau_1)\}\,ds - y_0 t\,.$$

By taking expectation in the latter identity and using that the process

$$z(t)\exp\{-i\tau - i\alpha(t+\tau_1)\}$$

is stationary, we get that

$$(P - i\alpha)u = y_0\,. \tag{1.79}$$

Here $u = \mathbf{E}(z(0)\exp\{-i\tau - i\alpha\tau_1\})$. Hence, equation (1.79) has a solution for any $y_0 \in \mathbf{C}^n$.

Let us show that it is unique. Indeed, let v be a solution of (1.79) distinct from u. It is then easy to show that the process $y(t) + \exp\{i(\alpha t + \tau)\}(u - v)$ is a periodic solution of system (1.77) distinct from $y(t)$, which contradicts the assumption of uniqueness of such a solution.

So, for arbitrary $y_0 \in \mathbf{C}^n$, equation (1.79) has a unique solution. Thus the points $i\alpha$, $\alpha \in \mathbf{R}$, do not belong to the spectrum of the matrix P and, hence, the spectrum of the monodromy matrix $X(T)$ does not intersect the imaginary axis. This proves necessity.

Sufficiency. Let $G(t,\tau)$ denote Green's function constructed from the matriciant of the linear system (1.74) [143, p. 153]. By the conditions of the theorem, $G(t,\tau)$ admits the estimate

$$||G(t,\tau)|| \le K\exp\{-\gamma|t-\tau|\}\,. \tag{1.80}$$

Using this matrix, define the function

$$y(t) = \int_{-\infty}^{\infty} G(t,\tau)\theta(\tau)\,d\tau + \sum_{i=-\infty}^{\infty} G(t, t_i + 0)\zeta_i\,. \tag{1.81}$$

Similarly to [143, p. 154] one can show that

$$\int_{-\infty}^{\infty} K\exp\{-\gamma|t-\tau|\}\mathbf{E}|\theta(\tau)|\,d\tau + \sum_{i=-\infty}^{\infty} K\exp\{-\gamma|t-\tau_i|\}\mathbf{E}|\zeta_i| < \infty\,.$$

Then, it follows from the Fubini theorem and the B. Levy theorem that the following integral and the series converge with probability 1,

$$\int_{-\infty}^{\infty} \|G(t,\tau)\|\,|\theta(\tau)|\,d\tau\,,$$

$$\sum_{i=-\infty}^{\infty} \|G(t,t_i+0)\|\,|\zeta_i|\,.$$

Hence, the right-hand side in (1.81) is defined with probability 1.

Using properties of Green's function it is easy to show that the integrals and the sums in (1.81) obtained by formally differentiating both sides with respect to t converge uniformly in t on any bounded line segment with probability 1.

Indeed, differentiating the right-hand side of (1.81) with respect to t we get an integral that, for arbitrary $t_0 > 0$, admits the following estimate for $n > t_0$:

$$\sup_{t\in[-t_0,\ t_0]}\left(\int_{-\infty}^{-n} K\exp\{-\gamma(t-\tau)\}|\theta(\tau)|\}\,d\tau\right)+\left(\int_{n}^{\infty} K\exp\{-\gamma(\tau-t)\}|\theta(\tau)|\,d\tau\right)$$
$$\le \exp\{\gamma t_0\}K\int_{-\infty}^{-n}\exp\{\gamma\tau\}|\theta(\tau)|\,d\tau+\exp\{\gamma t_0\}K\left(\int_{n}^{\infty}\exp\{-\gamma\tau\}|\theta(\tau)|\,d\tau\to 0\right.$$

with probability 1 for $n\to\infty$, since these integrals converge. Similarly one estimates remainders of the seres in (1.81).

Let us show that $y(t)$ defined by (1.81) is a solution of system (1.77). Indeed, for $t\ne t_i$, we have

$$\frac{dy}{dt}=\int_{-\infty}^{t}\frac{dG(t,\tau)}{dt}\theta(\tau)\,d\tau+\sum_{t_i<t}\frac{dG(t,t_i+0}{dt}\zeta_i+\int_{t}^{\infty}\frac{dG(t,\tau)}{dt}\theta(\tau)\,d\tau$$
$$+\sum_{t_i\ge t}\frac{dG(t,t_i+0}{dt}\zeta_i+[G(t,t-0)-G(t,t+0)]\theta(t)=Py(t)+\theta(t)\,.$$

For $t=t_j$, we get

$$\begin{aligned} y(t_j+0)-y(t_j) &= \int_{-\infty}^{\infty}[G(t_j+0,\tau)-G(t_j-0,\tau)]\theta(\tau)\,d\tau \\ &\quad+\sum_{t_i<t_j+0}G(t_j+0,t_i+0)\zeta_i+\sum_{t_i\ge t_j+0}G(t_j+0,t_i+0)\zeta_i \\ &\quad-\sum_{t_i<t_j-0}G(t_j-0,t_i+0)\zeta_i+\sum_{t_i\ge t_j-0}G(t_j-0,t_i+0)\zeta_i \\ &= G(t_j+0,t_j+0)\zeta_i+\sum_{t_i<t_j}[G(t_j+0,t_i+0)-G(t_j-0,t_i+0)]\zeta_i \end{aligned}$$

$$- G(t_j - 0, t_j + 0)\zeta_j + \sum_{t_i > t_j} [G(t_j + 0, t_i + 0)$$
$$- G(t_j - 0, t_i + 0)]\zeta_i = \zeta_j \,,$$

since the jumps of Green's function at the points $t_i \neq t_j$ are equal to 0. It is clear from (1.71) that $y(t)$ satisfies the estimate

$$\sup_{t \in \mathbf{R}} \mathbf{E}|y(t)| < \infty \,. \tag{1.82}$$

Trajectory-wise uniqueness of such a solution bounded in mean follows, since the difference of two solutions, which is mean bounded on the whole axis, is a solution, which is mean bounded on the whole axis, of homogeneous system (1.74) which, by conditions on the matrix P, has only the trivial solution that is bounded on $\mathbf{R}$. Thus, system (1.77) has a unique solution $y(t)$, mean bounded on $\mathbf{R}$, and, hence, it has a T-periodic solution $y^*(t)$ that is periodically connected with $\theta(t)$ and ζ_i. Its construction shows that $\mathbf{E}|y^*(0)|$ exists, and using the representation in [143, p. 16],

$$y^*(t) = y^*(0) + \int_0^t (Py^*(s) + \theta(s))\, ds + \sum_{0 < t_i < t} \zeta_i \,,$$

and an analogue of Gronwall-Bellman inequality for differentials and sums we get the estimate

$$\sup_{t \in [0,\ T]} \mathbf{E}|y^*(t)| < \infty \,.$$

This, because $y^*(t)$ is periodic, means that it is mean bounded on the whole axis. Uniqueness of such a solution implies the statement of the theorem. $\square$

Remark 1. If spectrum of the monodromy matrix $X(T)$ lies inside the unit circle, then the solution $y(t)$ in (1.81) has the form

$$y(t) = \int_{-\infty}^t \Phi(t,\tau)\theta(t)\, d\tau + \sum_{t_i < t} \Phi(t, t_i)\zeta_i,$$

where $\Phi(t, \tau)$ is the matriciant of system (1.74), and it admits the estimate

$$\|\Phi(t,\tau)\| \leq K \exp\{-\gamma(t - \tau)\}$$

for $t \geq \tau$. This easily yields exponential stability in the mean for a periodic solution of system (1.77), implying that for the periodic solution of system (1.70).

1.8 Weakly nonlinear systems

In this section, we give conditions for existence of periodic solutions of nonlinear impulsive systems that are close to linear ones.

Let system (1.1), which satisfies conditions 1)–3) of Section 1.5, have the form

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + \varepsilon f(t, x, \xi(t)), \qquad t \neq t_i, \\ \Delta x|_{t=t_i} &= B_i x + \varepsilon I_i(x, \eta_i)\,, \end{aligned} \tag{1.83}$$

where ε is a small positive parameter. Let also its right-hand sides be such that solutions of the Cauchy problem $x(t_0) = x_0(\omega)$ exist and trajectory-wise unique for $t \in \mathbf{R}$, $t_0 \in \mathbf{R}$, and $x_0(\omega)$ being an arbitrary random variable taking values in $\mathbf{R}^n$. To make calculations simpler, we assume that f and I_i are Lipschitz continuous with respect to x in $\mathbf{R}^n$.

Assume that

$$|f(t, 0, y)| \leq C(1 + |y|), \ |I_i(0, z)| \leq C(1 + |z|) \tag{1.84}$$

for some constant $C > 0$.

Theorem 1.13. *If spectrum of the monodromy matrix of system* (1.72) *does not intersect the unit disk, then for sufficiently small ε, any random process $\xi(t)$, and a sequence of random variables η_i satisfying conditions of Theorem 1.12, system* (1.83) *has a unique periodic solution $x(t)$ such that*

$$\sup_{0 \leq t \leq T} \mathbf{E}|x(t)| < \infty\,.$$

This solution is exponentially totally mean stable if spectrum of the monodromy matrix $X(T)$ of system (1.72) *lies inside the unit circle.*

Proof. Let us again apply the Lyapunov change of variables (1.75) to system (1.83). For $t \neq t_i$, we get

$$\begin{aligned} \frac{dx}{dt} &= \frac{dX(t)}{dt} \exp\{-Pt\}y - X(t)\exp\{-Pt\}Py + \Phi(t)\frac{dy}{dt} \\ &= A(t)X(t)\exp\{-Pt\}y + \varepsilon f(t, \Phi(t)y, \xi(t))\,. \end{aligned}$$

Whence,

$$\frac{dy}{dt} = Py + \varepsilon \Phi^{-1}(t) f(t, \Phi(t)y, \xi(t))\,.$$

For $t = t_i$, we have

$$x(t_i + 0) - x(t_i) = X(t_i + 0)\exp\{-Pt_i\}y(t_i + 0) - X(t_i)\exp\{-Pt_i\}y(t_i)$$
$$= B_i X(t_i)\exp\{-Pt_i\}y(t_i) + \varepsilon I_i(X(t_i)\exp\{-Pt_i\}y(t_i), \eta_i).$$

Or

$$X(t_i + 0)\exp\{-Pt_i\}\Delta y|_{t=t_i} = \varepsilon I_i(X(t_i)\exp\{-Pt_i\}y(t_i), \eta_i).$$

Whence we get

$$\Delta y|_{t=t_i} = \varepsilon \Phi^{-1}(t_i + 0) I_i(\Phi(t_i)y(t_i), \eta_i).$$

We thus finally obtain

$$\begin{aligned} \frac{dy}{dt} &= Py + \varepsilon g(t, y, \xi(t)), \qquad t \neq t_i, \\ \Delta y|_{t=t_i} &= \varepsilon J_i(y, \eta_i). \end{aligned} \tag{1.85}$$

with the functions g and J_i periodic in t and i. Since Φ and Φ^{-1} are bounded, g and J_i are Lipschitz continuous in y and satisfy inequality (1.84) with some Lipschitz constant L_1 and the constant C_1 from inequality (1.84).

Thus, as in the previous theorem, the problem of studying periodic solutions of system (1.83) is reduced to the same for system (1.85), with spectrum of the constant matrix P not intersecting the imaginary axis.

Let again $G(t, \tau)$ be Green's function for system (1.74) satisfying estimate (1.80). Define $y_1(t)$ to be a unique solution of the system

$$\begin{aligned} \frac{dy_1}{dt} &= Py_1 + \varepsilon g(t, 0, \xi(t)), \qquad t \neq t_i, \\ \Delta y_1|_{t=t_i} &= \varepsilon J_i(0, \eta_i). \end{aligned}$$

Theorem 1.12 gives existence and uniqueness of such a solution that satisfies the estimate

$$\sup_{t \in [0,\ T]} \mathbf{E}|y_1(t)| < \infty, \tag{1.86}$$

and it also gives the representation

$$y_1(t) = \varepsilon \int_{-\infty}^{\infty} G(t, \tau) g(\tau, 0, \xi(\tau))\, d\tau + \varepsilon \sum_{i=-\infty}^{\infty} G(t, t_i + 0) J_i(0, \eta_i). \tag{1.87}$$

The integral and the sum in (1.87) exist with probability 1 by estimates (1.80) and (1.84).

Define now a sequence of random processes

$$\{y_n(t):\ t \in \mathbf{R}\}, \qquad n \geq 2,$$

such that each member is a solution, periodic with period T, of the system

$$\begin{gathered} \frac{dy_n}{dt} = Py_n + \varepsilon g(t, y_{n-1}(t), \xi(t)), \qquad t \neq t_i\,, \\ \Delta y_n|_{t=t_i} = \varepsilon J_i(y_{n-1}(t_i), \eta_i)\,. \end{gathered} \tag{1.88}$$

We also have the following representation:

$$y_n(t) = \varepsilon \int_{-\infty}^{\infty} G(t,\tau) g(\tau, y_{n-1}(\tau), \xi(\tau))\, d\tau + \varepsilon \sum_{i=-\infty}^{\infty} G(t, t_i + 0) J_i(y_{n-1}(t_i), \eta_i) \tag{1.89}$$

and the estimate

$$\sup_{t\in[0,\ T]} \mathbf{E}|y_n(t)| < \infty\,. \tag{1.90}$$

But since the processes $y_n(t)$, being solutions of system (1.88), are left continuous with probability 1, they are also measurable. Using the Lipschitz condition, (1.84) and (1.90) we see that the process $g(t, y_{n-1}(t), \xi(t))$ satisfies condition (1.71). Using (1.89), as in [41, p. 214], we get

$$\begin{aligned} \mathbf{E}|y_n(t)| \leq \varepsilon \bigg(& \int_{-\infty}^{\infty} K \exp\{-\gamma|t-\tau|\} L_1 \mathbf{E}|y_{n-1}(\tau)|\, d\tau \\ & + \int_{-\infty}^{\infty} K \exp\{-\gamma|t-\tau|\} C_1 (1 + \mathbf{E}|\xi(\tau)|)\, d\tau \\ & + \sum_{i=-\infty}^{\infty} K \exp\{-\gamma|t-t_i|\} L_1 \mathbf{E}|y_{n-1}(t_i)| \\ & + \sum_{i=-\infty}^{\infty} K \exp\{-\gamma|t-t_i|\} C_1 (1 + \mathbf{E}|\eta_i|) \bigg) \\ \leq \varepsilon & \int_{-\infty}^{\infty} K \exp\{-\gamma|t-\tau|\} C_1 (1 + \mathbf{E}|\xi(\tau)|)\, d\tau \\ & + \varepsilon \sum_{i=-\infty}^{\infty} K \exp\{-\gamma|t-t_i|\} C_1 (1 + \mathbf{E}|\eta_i|) + \sup_{t\in[0,\ T]} \mathbf{E}|y_{n-1}(t)| \frac{2\varepsilon K L_1}{\gamma} \\ & + \sup_{t\in[0,\ T]} \varepsilon \mathbf{E}|y_{n-1}(t)| \frac{2\exp\{\gamma T(1-\frac{1}{p})\}}{1 - \exp\{-\frac{\gamma T}{p}\}}. \end{aligned}$$

The last estimate is obtained in a way similar to [143, p. 238]. Hence,

$$\mathbf{E}|y_n(t)| \le \varepsilon\Bigg(\int_{-\infty}^{\infty} K\exp\{-\gamma|t-\tau|\}C_1(1+\mathbf{E}|\xi(\tau)|)\,d\tau$$

$$+\sum_{i=-\infty}^{\infty} K\exp\{-\gamma|t-t_i|\}C_1(1+\mathbf{E}|\eta_i|)\Bigg)$$

$$+\,\varepsilon\left(\frac{2KL_1}{\gamma}+\frac{2\exp\{\gamma T(1-\frac{1}{p})\}}{1-\exp\{-\frac{\gamma T}{p}\}}\right)\sup_{t\in[0,\ T]}\mathbf{E}|y_{n-1}(t)|\,. \quad (1.91)$$

Choose ε such that the coefficient in the last term in (1.91) be less than 1. Then, continuing estimate (1.91) we get that

$$\sup_{n\ge 1}\sup_{t\in[0,\ T]}\mathbf{E}|y_n(t)| < \infty,$$

that is, all moments of the processes $y_n(t)$ are uniformly bounded. Set

$$r_n(t) = |y_n(t) - y_{n-1}(t)|.$$

We have

$$\mathbf{E}r_n(t) \le \varepsilon KL_1\int_{-\infty}^{\infty}\exp\{-\gamma|t-\tau|\}\mathbf{E}r_{n-1}(\tau)\,d\tau$$

$$+\,\varepsilon KL_1\sum_{i=-\infty}^{\infty}\exp\{-\gamma|t-t_i|\}\mathbf{E}r_{n-1}(t_i)$$

$$\le \sup_{t\in[0,\ T]}\mathbf{E}r_n(t)\left(\varepsilon\frac{2KL_1}{\gamma}+\varepsilon\frac{2KL_1\exp\{\gamma T(1-\frac{1}{p})\}}{1-\exp\frac{-\gamma T}{p}}\right). \quad (1.92)$$

The choice for ε gives mean convergence of $y_n(t)$ with probability 1, as $n\to\infty$, to some T-periodic random process $y(t)$ for every $t\in\mathbf{R}$. Moreover, Fatou's lemma gives that

$$\sup_{t\in[0,\ T]}\mathbf{E}|y(t)| < \infty\,.$$

Let us now show that convergence with probability 1 of $y_n(t)$ to $y(t)$ as $n\to\infty$ is uniform on every bounded line segment $[-t_0, t_0]$ of the real axis. We have

$$r_n(t) = |y_n(t) - y_{n-1}(t)| \le \varepsilon K L_1 (\exp\{\gamma t_0\} \int_{-\infty}^{-t_0} \exp\{\gamma\tau\} r_{n-1}(\tau)\, d\tau$$

$$+ \exp\{-\gamma t_0\} \int_{-t_0}^{0} \exp\{-\gamma\tau\} r_{n-1}(\tau)\, d\tau + \exp\{-\gamma t_0\} \int_{0}^{t_0} \exp\{\gamma\tau\} r_{n-1}(\tau)\, d\tau$$

$$+ \exp\{\gamma t_0\} \int_{t_0}^{\infty} \exp\{-\gamma\tau\} r_{n-1}(\tau)\, d\tau + \exp\{\gamma t_0\} \sum_{t_i < -t_0} \exp\{\gamma t_i\} r_{n-1}(t_i)$$

$$+ \exp\{-\gamma t_0\} \sum_{-t_0 \le t_i < 0} \exp\{-\gamma t_i\} r_{n-1}(t_i) + \exp\{-\gamma t_0\} \sum_{0 < t_i < t_0} \exp\{\gamma t_i\} r_{n-1}(t_i)$$

$$+ \exp\{\gamma t_0\} \sum_{t_i \ge -t_0} \exp\{-\gamma t_i\} r_{n-1}(t_i))$$

$$\le \sup_{t \in [-t_0,\ t_0]} r_{n-1}(t) \varepsilon A L_1 \left(\frac{2}{\gamma} + \frac{2\exp\{\gamma T (1 - \frac{1}{p})\}}{1 - \exp\{-\gamma \frac{T}{p}\}} \right),$$

where A is a constant.

Since the quantity

$$\sup_{[-t_0,\ t_0]} r_n(t)$$

is finite with probability 1 for every n, the choice of ε shows that

$$\sup_{[-t_0,\ t_0]} r_n(t) \to 0,\ n \to \infty,$$

with probability 1.

Passing to the limit in (1.89) as $n \to \infty$, since g and J_i are continuous with respect to y, we see that the limit process $y(t)$ satisfies the relation

$$y(t) = \varepsilon \int_{-\infty}^{\infty} G(t,\tau) g(\tau, y(\tau), \xi(\tau))\, d\tau + \varepsilon \sum_{i=-\infty}^{\infty} G(t, t_i + 0) J_i(y(t_i), \eta_i)\,. \quad (1.93)$$

Differentiating $y(t)$ with respect to t for $t \ne t_i$ and then calculating the value of the jump of the function $y(t)$ at $t = t_i$, as in Theorem 1.12, we see that $y(t)$ is a solution of system (1.85).

Let us now prove that this solution is unique. Let $z(t)$ be another T-periodic solution of (1.85) such that its first moment is bounded. We will show that $z(t)$ satisfies the integral-sum equation (1.93).

Indeed, since $z(t)$ is smooth in t on the intervals $(t_i,\ t_{i+1}]$ with probability 1 and satisfies the differential equation (1.85) and the jump conditions at t_i, for $\tau \in (t_i,\ t_{i+1}]$ we have

$$\frac{dz(\tau)}{d\tau} = Pz(\tau) + \varepsilon g(\tau, z(\tau)\xi(\tau))\,. \tag{1.94}$$

Multiplying this relation by $G(t,\tau)$ and integrating over $[t_i, t_{t+i}]$, the left-hand side becomes

$$\begin{aligned}
&\int_{t_i}^{t} G(t,\tau)\,dz(\tau) + \int_{t}^{t_{i+1}} G(t,\tau)\,dz(\tau) = G(t,t-0)z(t) \\
&\quad - G(t,t_i+0)z(t_i+0) - \int_{t_i}^{t} \frac{d}{d\tau}G(t,\tau)z(\tau)\,d\tau + G(t,t_{i+1}-0)z(t_{i+1}-0) \\
&\quad - G(t,t+0)z(t) - \int_{t}^{t_{i+1}} \frac{d}{d\tau}G(t,\tau)z(\tau)\,d(\tau) = [G(t,t-0)-G(t,t+0)]z(t) \\
&\quad - G(t,t_i+0)z(t_i+0) + G(t,t_{i+1}-0)z(t_{i+1}-0) - \int_{t_i}^{t_{i+1}} \frac{d}{d\tau}G(t,\tau)z(\tau)\,d\tau\,.
\end{aligned} \tag{1.95}$$

But, since

$$[G(t,t-0)-G(t,t+0)] = E\,,$$

and

$$\frac{dG(t,\tau)}{d\tau} = -PG(t,\tau)\,,$$

identity (1.95) becomes

$$z(t) - G(t,t_i+0)z(t_i+0) + G(t,t_{i+1}-0) + \int_{t_i}^{t_{i+1}} G(t,\tau)Pz(\tau)\,d\tau\,.$$

Comparing this expression with the one obtained in the right-hand side of (1.94) we get

$$\begin{aligned}
z(t) = \varepsilon &\int_{t_i}^{t_{i+1}} G(t,\tau)g(\tau,z(\tau),\xi(\tau))\,d\tau + G(t,t_i+0)z(t_i+0) \\
&- G(t,t_{i+1}-0)z(t_{i+1}-0)\,.
\end{aligned} \tag{1.96}$$

Making the same reasoning for the segment $(t_i,\ t_{i+2}]$, since $z(t)$ is piecewise smooth, for $t \neq t_{i+1}$ we get

$$[G(t,t-0)-G(t,t+0)]z(t)-G(t,t_i+0)z(t_i+0)+G(t,t_{i+1}-0)z(t_{i+1}-0)$$
$$-G(t,t_{i+1}+0)z(t_{i+1}+0)+G(t,t_{i+2}-0)z(t_{i+2}-0)+\int_{t_i}^{t_{i+2}}G(t,\tau)Pz(\tau)\,d\tau$$
$$=z(t)-G(t,t_i+0)z(t_i+0)+G(t,t_{i+1}-0)z(t_{i+1}-0)$$
$$-G(t,t_{i+1}+0)z(t_{i+1}-0)-\varepsilon G(t,t_{i+1}+0)J_{i+1}(z(t_{i+1}-0),\eta_{i+1})$$
$$+G(t,t_{i+2}-0)z(t_{i+2}-0)+\int_{t_i}^{t_{i+2}}G(t,\tau)Pz(\tau)\,d\tau=z(t)$$
$$-G(t,t_i+0)z(t_i+0)-\varepsilon G(t,t_{i+1}+0)J_{i+1}(z(t_{i+1}-0),\eta_{i+1})$$
$$+\int_{t_i}^{t_{i+2}}G(t,\tau)Pz(\tau)\,d\tau+G(t,t_{i+2}-0)z(t_{i+2}-0).$$

We have used above that the function $G(t,\tau)$ is smooth for $t\neq\tau$.

If $t=t_{i+1}$, then the condition imposed on the jump gives

$$G(t_{i+1},t_{i+1}-0)z(t_{i+1}-0)-G(t_{i+1},t_i+0)z(t_i+0)$$
$$-G(t_{i+1},t_{i+1}+0)z(t_{i+1}+0)+G(t_{i+1},t_{i+2}-0)z(t_{i+2}-0)$$
$$+\int_{t_i}^{t_{i+2}}G(t_{i+1},\tau)Pz(\tau)\,d\tau$$
$$=z(t_{i+1}-0)-\varepsilon G(t_{i+1},t_{i+1}+0)J_{i+1}(z(t_{i+1}-0),\eta_{i+1})$$
$$-G(t_{i+1},t_i+0)z(t_i+0)+G(t_{i+1},t_{i+2}-0)z(t_{i+2}-0)$$
$$+\int_{t_i}^{t_{i+2}}G(t_{i+1},\tau)Pz(\tau)\,d\tau\,.$$

Continuing this procedure on the intervals $(t_i,t_{i+j}]$ and $(t_{i-j},t_i]$, $j=3,4\ldots$, we get a representation of $z(t)$ for $t\in(t_{i-j},t_{i+j}]$,

$$z(t)=\varepsilon\int_{t_{i-j}}^{t_{i+j}}G(t,\tau)g(\tau,z(\tau),\xi(\tau)\,d\tau+\varepsilon\sum_{t_{i-j}\le t_k<t_{i+j}}G(t,t_k+0)J_k(z(t_k),\eta_k))$$
$$+\,G(t,t_{i-j}+0)z(t_{i-j}+0)-G(t,t_{i+j}-0)z(t_{i+j}-0)\,.$$

Using the last relation we have

$$\mathbf{E}|z(t)-\varepsilon\int_{-\infty}^{\infty}G(t,\tau)g(\tau,z(\tau),\xi(\tau))\,d\tau-\varepsilon\sum_{k=-\infty}^{\infty}G(t,t_k+0)J_k(z_(t_k),\eta_k)|$$
$$\le\mathbf{E}|\varepsilon\int_{-\infty}^{t_{i-j}}G(t,\tau)g(\tau,z(\tau),\xi(\tau))\,d\tau+\varepsilon\int_{t_{i+j}}^{\infty}G(t,\tau)g(\tau,z(\tau),\xi(\tau))\,d\tau$$

$$+\,\varepsilon \sum_{t_k<t_{i-j}} G(t,t_k+0)J_k(z(t_k),\eta_k)$$

$$+\,\varepsilon \sum_{t_k>t_{i+j}} G(t,t_k+0)J_k(z(t_k),\eta_k) + G(t,t_{i-j}+0)$$

$$\times\, z(t_{i-j}+0) - G(t,t_{i+j}-0)z(t_{i+j}-0)| \le \varepsilon \int_{-\infty}^{t_{i-j}} K\exp\{-\gamma|t-\tau|\}$$

$$\times\, \mathbf{E}|g(\tau,z(\tau),\xi(\tau))|\,d\tau + \varepsilon\int_{t_{i+j}}^{\infty} K\exp\{-\gamma|t-\tau|\}\mathbf{E}|g(\tau,z(\tau),\xi(\tau))|\,d\tau$$

$$+\,\varepsilon \sum_{t_k<t_{i-j}} K\exp\{-\gamma|t-t_k|\}\mathbf{E}|J_k(z(t_k),\eta_k)|$$

$$+\,\varepsilon \sum_{t_k>t_{i+j}} K\exp\{-\gamma|t-t_k|\}\mathbf{E}|J_k(z(t_k),\eta_k)|$$

$$+\,K\exp\{-\gamma|t-t_{i-j}|\}\mathbf{E}|z(t_{i-j}+0)| + K\exp\{-\gamma|t-t_{i+j}|\}\mathbf{E}|z(t_{i+j})|\,.$$

The above inequality holds for an arbitrary natural j. Making $j\to\infty$, since the integrals and the sums in the left-hand side are mean convergent and that $z(t)$ is mean bounded, we get that

$$\mathbf{E}|z(t) - \varepsilon\int_{-\infty}^{\infty} G(t,\tau)g(\tau,z(\tau),\xi(\tau))\,d\tau - \varepsilon\sum_{i=-\infty}^{\infty} G(t,t_i+0)J_i(y(t_i),\eta_i)| = 0$$

for arbitrary $t\in\mathbf{R}$ and, hence, for every real t the following holds with probability 1:

$$z(t) = \varepsilon\int_{-\infty}^{\infty} G(t,\tau)g(\tau,z(\tau),\xi(\tau))\,d\tau + \varepsilon\sum_{i=-\infty}^{\infty} G(t,t_i+0)J_i(y(t_i),\eta_i). \quad (1.97)$$

By subtracting (1.97) from (1.93) and taking expectations, we get

$$\mathbf{E}|y(t)-z(t)| \le \varepsilon\int_{-\infty}^{\infty} K\exp\{-\gamma|t-\tau\}\mathbf{E}|y(\tau)-z(\tau)|\,d\tau$$

$$+\,\varepsilon K\sum_{i=-\infty}^{\infty}\exp\{-\gamma|t-t_i|\}\mathbf{E}|y(t_i)-z(t_i)| \le \sup_{t\in\mathbf{R}}\mathbf{E}|z(t)-y(t)|$$

$$\times\left(\varepsilon\frac{2KL_1}{\gamma} + \varepsilon\frac{2KL_1\exp\{\gamma T(1-\frac{1}{p})\}}{1-\exp\{-\gamma\frac{T}{p}\}}\right).$$

Since ε is chosen so that the quantity in the parentheses would be less than 1 and

$$\sup_{t\in\mathbf{R}} \mathbf{E}|y(t) - z(t)| \le \sup_{t\in\mathbf{R}} \mathbf{E}|y(t)| + \sup_{t\in\mathbf{R}} \mathbf{E}|z(t)| < \infty\,,$$

for every $t \in \mathbf{R}$, we have

$$\mathbf{E}|y(t) - z(t)| = 0\,,$$

which precisely means that a periodic solution is unique up to stochastic equivalence.

Let us now prove that the obtained solution is totally exponentially mean stable in the case where the multipliers of system (1.72) lie inside the unit circle. In this case, eigen values of the matrix P have negative real parts and Green's function has the form

$$G(t,\tau) = \exp\{P(t-\tau)\} = \Phi(t,\tau)$$

and admits the estimate

$$||G(t,\tau)|| \le K\exp\{-\gamma(t-\tau)\},\ t \ge \tau\,, \tag{1.98}$$

with some positive constants K and γ that are independent of t and τ. Let $x(t)$ be an arbitrary solution of system (1.85) such that $x(0,x_0) = x(\omega)$ and $M|x_0(\omega)| < \infty$.

It follows from [143, p. 241] that $x(t,x_0)$ admits the representation

$$\begin{aligned} x(t,x_0) = \Phi(t,0)x_0 &+ \varepsilon\int_0^t \Phi(t,\tau)g(\tau,x(\tau,x_0),\xi(\tau)\,d\tau \\ &+ \varepsilon\sum_{0<t_i<t}\Phi(t,t_i)J_i(x(t_i,x_0)\eta_i)\,. \end{aligned}$$

Using (1.98) we get from the above relation that

$$\begin{aligned} \mathbf{E}|y(t) - x(t,x_0)| \le K\exp\{-\gamma t\}&\mathbf{E}|y(0) - x_0| \\ &+ \varepsilon\int_0^t K\exp\{-\gamma(t-\tau)\}L_1\mathbf{E}|y(\tau) - x(\tau,x_0)|\,d\tau \\ &+ \varepsilon\sum_{0<t_i<t} K\exp\{-\gamma(t-t_i)\}L_1\mathbf{E}|y(t_i) - x(t_i,x_0)| \end{aligned}$$

or

$$u(t) \le K\mathbf{E}|y(0) - x_0| + \varepsilon\int_0^t KL_1u(\tau)\,d\tau + \varepsilon\sum_{0<t_i<t} KL_1u(t_i)\,,$$

where $u(t) = \exp\{\gamma t\}M|x(t,x_0) - y(t)|$.

With a use of the generalized Gronwall-Bellman lemma, the last inequality gives

$$u(t) \le K\mathbf{E}|x_0 - y_0| \exp\{\varepsilon K L_1 t\}(1 + \varepsilon K L_1)^{i(0,t)}, \tag{1.99}$$

where $i(0, t)$ is the number of impulses on the interval $(0, t)$.

However, since $i(0, t) \le p + \frac{p}{T}t$, we see that

$$u(t) \le K_1 \exp\{(\varepsilon K L_1 + \frac{p}{T} \ln(1 + \varepsilon K L_1))t\}\mathbf{E}|x_0 - y(0)|,$$

where $K_1 = K(1 + \varepsilon K L_1)$, whence we finally get that

$$\mathbf{E}|x(t, x_0) - y(t)| \le K_1 \exp\{-\gamma + N_1(\varepsilon))t\}\mathbf{E}|x_0 - y(0)|, \tag{1.100}$$

for all $t \ge 0$, $N_1(\varepsilon) = \varepsilon K L_1 + \frac{p}{T} \ln(1 + \varepsilon K L_1)$. If we require that $N_1(\varepsilon) < \gamma$, then (1.100) yields that $y(t)$ is totally asymptotically stable with exponentially decay. □

Remark 1. Similarly to [41, p. 218], a solution of system (1.85) will be called asymptotically periodic in the mean.

Remark 2. A result similar to Theorem 1.13 for systems without impulsive effects was obtained in [41, p. 222] by using another method. However, as opposed to Theorem 1.13, there is only a proof for existence of a periodic solution. The method used in the proof of Theorem 1.13 permitted to also prove uniqueness and stability of such a solution.

1.9 Comments and References

Section 1.1. Differential equations with impulsive effects have appeared at the dawn of a study of nonlinear mechanics as an apparatus that permits to adequately describe oscillation processes influenced by forces of small duration. A well known example that leads to such an equation is a model of a pendulum watch clock proposed in 1937 by M. M. Krylov and M. M. Bogolyubov in [86]. Using this example it has been shown that asymptotic methods of nonlinear mechanics can be successfully applied to a study of impulsive systems.

A subsequent development of this theory is due to works of Yu. A. Mitropol'sky, A. D. Myshkis, N. A. Perestyuk, and other authors. A use of a jump for solutions of impulsive systems, in place of the δ-function, was proposed by A. D. Myshkis and A. M. Samoilenko, see e.g. [117] and [136], which made a basis for the averaging principle and permitted to classify such systems. Qualitative theory for the behavior of solutions of impulsive systems is developed in works of Yu. A. Mitropol'sky, N. A. Perestyuk, O. S. Chernikova [105], Yu. A. Mitropol'sky, A. M. Samoilenko, N. A. Perestyuk [109], and others. A. M. Samoilenko and N. A. Perestyuk's monograph [143] is a result of numerous studies in this field.

However, in the works cited above, the impulsive systems, which are corresponding mathematical models of real processes, do not take into account the influence of random forces, and this leads to an incomplete description of such systems. This reveals a need to consider differential systems with random impulsive effects. Probably, the first work that deals with a system with impulsive effects is the book of V. D. Mil'man and A. D. Myshkis [102], where the authors study the limit behavior, as $t \to \infty$, of solutions of a linear impulsive system with random impulses at fixed times. Among other works in this direction, let us mention the works by J. M. Bismut [18], N. Karoni [64], H. Kushner [90], J. M. Lepeltier and B. Marchal [91], H. Nagai [118], which deal with optimal control for systems with impulsive effects at random times.

Sections 1.2—1.4. The limit behavior of solutions of impulsive systems, boundedness of solutions in the probability sense are considered in a number of publications of Ye. F. Tsarkov and his collaborators [190, 188, 189, 191, 179]. Related results are obtained by V. V. Anisimov in [4, 5, 3], and V. S. Korolyuk in [78, 79]. Limit behavior of semi-Markov processes with switchings were studied by A. V. Svishchuk [180].

However, in the mentioned works, the right-hand sides of the systems, the values of the impulses and their times were assumed to satisfy the conditions that implied that the solutions of the impulsive systems would be Markov or semi-Markov processes, which permitted to study them with analytical methods used in the theory of Markov processes, and these methods are not very "sensible" to the fact that the trajectories of the solutions are discontinuous. At the same time, the values of the impulses and times of the impulsive effects are not always Markov or semi-Markov in systems that model real processes. So the results obtained in the mentioned works can not be applied to such systems.

Moreover, these works, as a rule, deal with problems that are more typical for the theory of random processes, mainly obtaining limit theorems for distributions, and contain few qualitative results about differential equations with random right-hand sides, namely, existence of solutions on a maximal interval, conditions permitting to extend them over an infinite interval, their dissipativity and stability.

The results given in these sections are obtained by O. M. Stanzhyts'kyi and V. V. Ishchuk [172], and O. M. Stanzhyts'kyi [164].

Sections 1.5—1.6. Periodic in the restricted sense solutions of impulsive systems correspond, in practice, to periodic operation modes of real objects. Questions related to existence of such solutions of differential equations with random right-hand sides without impulsive effects were first studied by R. Z. Khasminskii in [70, p. 76] and [71]. Results in [70] were generalized by K. Ito and M. Nisio [62], and extended to equations with delay by V. B. Kolmanovsky [73].

The results discussed in the above sections were obtained by N. A. Perestyuk and O. M. Stanzhyts'kyi in [122], and generalize those obtained in [70, p. 76] and [71] to systems with impulses.

Sections 1.7—1.8. A. A. Dorogovtsev in [41] have obtained conditions, different from those in [70], for existence of periodic solutions for linear and weakly nonlinear differential systems with random right-hand sides by using Green's function for the linear part of the system. The same type results for impulsive deterministic systems were obtained by A. M. Samoilenko and N. A. Perestyuk in [143]

The results of these sections were obtained by the authors and N. A. Perestyuk in [142].

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Chapter 2

Invariant sets for systems with random perturbations

This chapter deals with the most important problem in the qualitative theory of differential equations, namely, a study of invariant manifolds for a system of differential equations with random control perturbations and stochastic Ito-type systems.

In Section 2.1, we introduce a notion of an invariant set for a differential systems with regular random perturbations. We obtain conditions for such sets to be zeros of a Lyapunov function of a reduced deterministic system and study stability of these sets. Local invariant sets have also been studied.

Section 2.2 studies invariant sets for stochastic Ito systems. Sufficient conditions for invariance of the zero set for a nonnegative Lyapunov function are obtained in terms of the Lyapunov function and a generating operator of the corresponding Markov process. We also study stochastic stability of the invariant set.

In Section 2.3, we consider the behavior of invariant sets when a small perturbation is applied to both a regular random perturbation and "white noise" type perturbations. It is shown that if the unperturbed deterministic system has a compact asymptotically stable manifold, then the perturbed system has an invariant manifold too.

Sections 2.4 and 2.5 generalize the Pliss reduction principle, known in the stability theory, to an equation with random perturbations and stochastic Ito systems. Here we show that the stability problem for equilibriums of a stochastic system can be reduced to a study of stability of a deterministic system on some manifold.

In Sections 2.6–2.7, we apply the reduction principle to study invariant sets for stochastic systems with both regular perturbations and "white noise" type perturbations. This is done by studying stability of a deterministic system on some manifolds.

2.1 Invariant sets for systems with random right-hand sides

In this section, we will study invariant sets for the following differential systems with random perturbations:

$$\frac{dx}{dt} = G(t, x, \xi(t)), \tag{2.1}$$

where $\xi(t)$ is a random process. Denote by S a Borel subset of $\{t \geq 0\} \times \mathbf{R}^n$. Let S_t be a subset of $\mathbf{R}^n$, where $S_t = \{x : (t, x) \in S\}$, and S_t is nonempty for all $t \geq 0$.

Definition 2.1. A set S is called *positively invariant with probability* 1 for system (2.1) if

$$\mathbf{P}\{(t, x(t, t_0, x_0)) \in S,\ \forall t \geq 0\} = 1 \tag{2.2}$$

for $(t_0, x_0(\omega)) \in S$, where $x(t, t_0, x_0)$ is a solution of (2.1) such that $x(t_0, t_0, x_0) = x_0$, $t_0 \geq 0$.

Remark 1. In the sequel we will only consider continuous solutions of system (2.1), so, for continuous and closed sets S that are those described by continuous functions, identity (2.2) in Definition 2.1 is equivalent to the following:

$$\mathbf{P}\{(t, x(t, t_0, x_0)) \in S\} = 1 \qquad \forall\, t \geq 0\,. \tag{2.3}$$

Indeed, it follows from (2.2) that (2.3) holds. Identity (2.2) follows from (2.3) because of the following. Let $t_i \geq 0$ be a rational. Denote

$$A_i = \{\omega \in \Omega : (t_i,\ x(t_i, t_0, x_0)) \in S\}\,.$$

Identity (2.3) yields $\mathbf{P}(A_i) = 1$, hence $\mathbf{P}(\overline{A_i}) = 0$. This means that the probability of their countable union,

$$\bigcup_i \overline{A_i} = \{\omega : \exists i, (t_i,\ x(t_i, t_0, x_0)) \notin S\}\,,$$

equals zero. So, the probability of the set

$$\bigcap_i A_i = \{\omega : \ (t_i, \ x(t_i, t_0, x_0)) \in S, \ \forall t_i \geq 0\}$$

is 1. This shows that the trajectory of the process $(t, x(t, t_0, x_0, \omega_0))$, for $\omega_0 \in \cap_i A_i$, belongs to S for all $t \geq 0$, because otherwise there would exist a rational point t_i, due to continuity of the trajectories and the set S, such that $(t_i, x(t_i, t_0, x_0)) \notin S$. This is a contradiction.

Definition 2.2. A set S is called *stable in probability* if for all $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that

$$\mathbf{P}\{\rho(x(t, t_0, x_0), S_t) > \varepsilon_1\} < \varepsilon_2 \tag{2.4}$$

for arbitrary $t \geq t_0$ if $\rho(x_0, S_{t_0}) < \delta$.

Definition 2.3. A set S is called *asymptotically stable in probability* if it is stable in probability and for arbitrary $\varepsilon > 0$ there exists $r(\varepsilon)$ such that

$$\mathbf{P}\{\rho(x(t, t_0, x_0), S_t) > \varepsilon\} \to 0$$

for $t \to \infty$ whenever $\rho(x_0, S_{t_0}) < r$.

Definition 2.4. A set S is called *totally stable in probability* if it is stable in probability and for arbitrary x_0, $\varepsilon > 0$, and $\delta > 0$ there exists $T = T(x_0, \varepsilon, \delta)$ such that (2.4) holds for $t > T$.

It is clear that the problem of studying invariant sets is very complicated. We will be limited only to the case where the random perturbations enter the system linearly, that is, we will consider the system

$$\frac{dx}{dt} = F(t, x) + \sigma(t, x)\xi(t)\,, \tag{2.5}$$

where $\xi(t)$ is a separable random process, integrable with probability 1 on any bounded interval of the positive semiaxis and takes values in $\mathbf{R}^m$. The vector-valued function $F(t, x)$ and the $n \times m$-matrix $\sigma(t, x)$ are defined and Borel measurable with respect to the totality of their variables for $t \geq 0$ and $x \in \mathbf{R}^n$ and are such that (2.5) satisfies existence and trajectory-wise uniqueness conditions for a solution of the Cauchy problem for the initial conditions in a bounded domain $D \subset \mathbf{R}^n$.

We will express the conditions for existence of invariant sets for system (2.5) in terms of Lyapunov functions for the "truncated" deterministic system

$$\frac{dx}{dt} = F(t, x)\,. \tag{2.6}$$

Let a nonnegative function $V(t,x)$ be defined on $\mathbf{R}_+ \times \overline{D}$, absolutely continuous in t, and Lipschitz continuous in x with a constant B_D. Denote by N_0^t the set of zeros of this function in D for a fixed $t \geq 0$, and assume that this set is nonempty. We will also assume that $Pr_{\mathbf{R}^n} N_0^t$ is a compact subset of D (here $Pr_{\mathbf{R}^n} N_0^t$ is the projection of the set $\cup_{t\geq 0} N_0^t$ onto $\mathbf{R}^n$). Denote it by N_0.

Theorem 2.1. *Let the above conditions be satisfied. If for $x \in D$, $t \geq 0$, there exist positive constants C_1 and C_2 such that*

$$\frac{d^0 V}{dt} \leq C_1 V,\ \ ||\sigma(t,x)|| \leq C_2 V(t,x)\,, \tag{2.7}$$

where $\frac{d^0V}{dt}$ is the Lyapunov operator along (2.6)*, then the set*

$$V(t,x) = 0,\ \ x \in D\,, \tag{2.8}$$

is positively invariant for system (2.5)*.*

If the function $V(t,x)$ is defined for $x \in \mathbf{R}^n$ and is globally Lipschitz continuous in x with a constant B and

$$\inf_{t\geq 0,\ x:\rho(N_0^t,x)>\delta} V(t,x) = V_\delta > 0,\ \ \delta > 0,\ \ \frac{d^0V}{dt} \leq -C_1 V\,, \tag{2.9}$$

and the process $|\xi(t)|$ satisfies the law of large numbers and

$$\sup_{t\geq 0} \mathbf{E}|\xi(t)| < \frac{C_1}{BC_2}\,, \tag{2.10}$$

then the set N_0^t is totally asymptotically stable in probability.

Proof. Consider a solution $x(t,x_0)$ of system (2.5) such that $x(0,x_0) = x_0(\omega)$ belongs to the set N_0^0 of zeros of the function $V(t,x)$ for $t = 0$ with probability 1. Since N_0 is a compact subset of D, we have that N_0^0 is a compact subset of D and, hence, $x(t,x_0) \in D$ and so $\tau(\omega) > 0$ with probability 1 for t in some interval $[0,\ \tau(\omega))$, where $\tau(\omega)$ is a random variable. Hence, using (2.6) and a lemma in [70, p.28] we have

$$\begin{aligned}\frac{dV(t,x(t,x_0))}{dt} &\leq \frac{d^0V(t,x(t,x_0))}{dt} + B_D C_2 V(t,x(t,x_0)|\xi(t)|) \\ &\leq C_1 V(t,x(t,x_0)) + B_D C_2 |\xi(t)| V(t,x(t,x_0)) \end{aligned} \tag{2.11}$$

for almost all $t \in [0,\ \tau(\omega))$ with respect to the Lebesgue measure. Here $\frac{dV}{dt}$ is the derivative along system (2.5).

Now, using the inequality in Lemma 1.1 we get

$$V(t, x(t, x_0)) \leq V(0, x_0) \exp\left\{ \int_0^t (B_D C_2 |\xi(t)| + C_1)\, ds \right\} \tag{2.12}$$

for $t \in [0,\ \tau(\omega))$.

It follows from (2.12) that

$$V(t, x(t, x_0)) = 0 \tag{2.13}$$

for $t \in [0,\ \tau(\omega))$. This shows that $x(t, x_0) \in N_0^t$ with probability 1 for all $t \in [0, \tau(\omega))$. Since N_0 is a compact subset of D, the assumption that $\tau(\omega)$ is finite with positive probability contradicts (2.12) and (2.13). Hence, $\tau(\omega) = \infty$ with probability 1 and, since $x(t, x_0)$ is a separable process, N_0^t consists of curves $(t, x(t, x_0))$, where $x(t, x)$ is a solution of system (2.5) such that $x(0, x_0) = x_0(\omega) \in N_0^0$ with probability 1. This shows, by definition, that the set N_0^t is positively invariant, proving the first part of the theorem.

We now prove the second part. Let ε and γ be arbitrary positive numbers. Denote by $U_\varepsilon(N_0^t)$ an ε-neighborhood of the set N_0^t. Since N_0 is compact, we can assume that $U_\varepsilon(N_0^t) \subset \mathbf{R}_+ \times D$.

Using (2.9) as for (2.12) we get

$$V(t, x(t, x_0)) \leq V(t_0, x_0) \exp\left\{ \int_{t_0}^t (B_D C_2 |\xi(t)| - C_1)\, ds \right\} \tag{2.14}$$

for $t \geq t_0$.

Using (2.10) and the fact that $|\xi(t)|$ satisfies the law of large numbers we can choose $T > 0$ such that

$$\mathbf{P}\left\{ \frac{1}{t} \int_{t_0}^{t_0+t} |\xi(s)|\, ds > \frac{C_1}{BC_2} \right\} < \gamma \tag{2.15}$$

for $t \geq T$.

After that, choose $A > 1$ so large that

$$\mathbf{P}\left\{ BC_2 \int_{t_0}^{T} |\xi(s)|\, ds > \ln A \right\} < \gamma\,. \tag{2.16}$$

Finally, take a δ-neighborhood of the set $N_0^{t_0}$ such that

$$V(t_0, x_0) \leq \frac{V_\varepsilon}{A}$$

for arbitrary $x_0 \in U_\delta(N_0^{t_0})$. Applying a lemma in [139, p. 61] we see that such a choice for δ is always possible.

Considering separately the cases $t < T$ and $t \geq T$ we see that (2.14)–(2.16) imply the inequality

$$\mathbf{P}\left\{\rho(x(t,x_0), N_0^t) > \varepsilon\right\} \leq \mathbf{P}\left\{V(t, x(t,x_0)) \geq V_\varepsilon\right\} < \gamma$$

that holds for $x_0 \in U_\delta(N_0^{t_0})$ and all $t \geq t_0$. It remains to show that the set N_0^t is stable in probability.

To prove that it is totally asymptotically stable, we will use that

$$\mathbf{P}\left\{\frac{1}{t}\int_{t_0}^{t_0+t} |\xi(s)|\, ds > \frac{C_1}{BC_2}\right\} \to 0,\ t \to \infty.$$

So, $V(t, x(t,x_0)) \to 0$ for $t \to \infty$ in probability. This gives that $\rho(x(t,x_0), N_0^t) \to 0$ for $t \to \infty$ in probability. $\square$

Remark. The second inequality in (2.7) shows that the action of the process $\xi(t)$ vanishes on the invariant set $V(t,x) = 0$. This condition can be weakened although it can not be dropped altogether, which is seen in the following example.

Example 1. Let

$$\frac{dx}{dt} = -x + \xi(t),$$

where x is scalar-valued.

The function $V(x) = x^2$ satisfies all conditions of the theorem in a neighborhood of zero save for the second condition (2.7). It is clear that the set $x = 0$, $t \geq 0$, is not invariant for the considered equation.

Clearly, to use this theorem, one needs to find Lyapunov functions with necessary properties. This can be done in a number of cases.

Example 2. Consider a Liénard equation well known in radio engineering,

$$x'' + f(x)x' + g(x) = \sigma(x, x')\xi(t,\omega). \tag{2.17}$$

Here $\sigma(x,x')$ measures the density of the external random forces that act on the system. Let

$$0 < C_1 < \frac{g(x)}{x} < C_2,\ 0 < C_3 < f(x) < C_4$$

for some $x_0 > 0$ in the domain $|x| > x_0$. Denote

$$F(x) = \int_0^x f(t)\,dt,\ G(x) = \int_0^x g(t)\,dt,$$

$$W(x,y) = (F(x) - \gamma x)y + G(x) + \int_0^x f(t)(F(t) - \gamma t)\,dt + 1 + \frac{y^2}{2}.$$

Choose V to be

$$V(x,y) = \begin{cases} [W(x,y)]^\alpha - C, & [W(x,y)]^\alpha > C, \\ 0, & [W(x,y)]^\alpha \le C. \end{cases}$$

By appropriately choosing α, γ, and C, it is easy to show that $V(x,y)$ satisfies all conditions of the theorem — the first one in (2.9) follows, since $V(x,y) \to \infty$ for $(x^2+y^2)^{\frac{1}{2}} \to \infty$. Hence, if $\|\sigma(x,y)\| \le C_5 V(x,y)$, then the set $V(x,y) = 0$ is invariant for (2.17). If the random process $\xi(t)$ satisfies conditions of the theorem with the constants chosen independently, then this set is totally asymptotically stable in probability.

Let us now consider locally invariant sets for systems of type (2.1). We will need a corresponding definition. Let S be a closed subset of $\mathbf{R}^{n+1}$, nonempty for any $t \in \mathbf{R}$. Choose an arbitrary $t_0 \in \mathbf{R}$. Denote

$$\tau_+(t_0,x_0) = \inf_{t>t_0}\{(t, x(t,x_0)) \notin S\}$$

and

$$\tau_-(t_0,x_0) = \sup_{t<t_0}\{(t, x(t,x_0)) \notin S\}.$$

Definition 2.5. A set S is called *locally invariant* for system (2.1) if

$$\tau_-(t_0,x_0) < \tau_+(t_0,x_0)$$

for $(t_0,\ x_0(\omega)) \in S$.

Theorem 2.2. *If for system* (2.5) *there exists a nonnegative Lyapunov function $V(t,x)$ on $\mathbf{R} \times \overline{D}$ such that*

$$\frac{d^0 V}{dt} \le C_1 V,\ ||\sigma(t,x)|| \le C_2 V(t,x),$$

then the set

$$V(t,x) = 0,\ x \in D,$$

if it is nonempty, is a locally invariant set for system (2.5).

Proof. Let $(t_0, x_0(\omega))$ belong to the set of zeros of $V(t,x)$, $x \in D$, with probability 1. Hence, $x(t,x_0) \in D$ for $t \in [t_0,\ \tau(\omega))$. Similarly to Theorem 2.1, we have the following estimate on this interval:

$$V(t, x(t,x_0)) \le V(t_0, x_0) \exp\{\int_{t_0}^{t} (B_D C_2 |\xi(t)| + C_1)\, ds\},$$

which proves the theorem. □

2.2 Invariant sets for stochastic Ito systems

Let us now study invariant sets for stochastic Ito systems.

Consider the system

$$dx = \ a(t,x)dt \ + \sum_{r=1}^{k} b_r(t,x) dw_r(t) \tag{2.18}$$

where $t \ge 0$, $x \in \mathbf{R}^n$, $a(t,x)$ and $b_r(t,x)$ $(r = \overline{1,k})$ are vectors in $\mathbf{R}^n$, $w_1(t), \dots, w_r(t)$ are linearly independent scalar Wiener processes defined on some complete probability space (Ω, F, P). We will assume that the coefficients a and b_r are nonrandom and such that equation (2.18) has a unique strong solution for the Cauchy problem with the initial conditions $x(t_0) = x_0 \in \mathbf{R}^n$ for $t \ge t_0$. Conditions that imply existence and uniqueness of such a solution are well known, see e.g. [54, p. 281], they are satisfied, for example, for functions $a(t,x)$, $b_r(t,x)$ that are Borel in the totality of the variables, Lipschitz continuous with respect to x in the domain $\{t \ge 0\} \times \mathbf{R}^n$, and such that $a(t,0)$ and $b(t,0)$ are bounded.

Invariance will be understood in the sense of the preceding section, namely, in the sense of Definition 2.1 where we take $t \ge t_0$ in formula (2.2). Taking into account that solutions of the Ito system is a Markov process, we will study stability in the following stronger sense.

Definition 2.6. A set S is called *stochastically stable* if for all $\varepsilon_1 > \ 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that

$$\mathbf{P}\left\{ \sup_{t \ge t_0} \rho(x(t,t_0,x_0), S_t) > \varepsilon_1 \right\} < \varepsilon_2 \tag{2.19}$$

for $\rho(x_0, S_{t_0}) < \delta$.

Clearly, this definition is more restrictive than Definition 2.2. It means that not only the probability of the solution leaving the ε_1-neighborhood is small but also that the solution itself lies in this neighborhood with a probability close to one. Conditions for existence and stability of invariant sets for (2.18) will be given in terms of Lyapunov functions $V(t,x)$ similarly as in [139, Ch. 2] for deterministic systems.

Let D be a bounded domain in $\mathbf{R}^n$ and a nonnegative function $V(t,x)$ be defined on a domain $\{t \geq 0\} \times \overline{D}$ such that it is continuously differentiable in t and twice continuously differentiable in x. Let N be the set of its zeros in $\{t \geq 0\} \times D$. Denote by N_t the set of $x \in D$ such that $V(t,x) = 0$ for a fixed $t \geq 0$. We also assume that it is nonempty in D for arbitrary $t \geq 0$. Let also the projection on $\mathbf{R}^n$ of the set N of zeros of the function V be closed in D.

Conditions for positive invariance and stochastic stability of the set $V(t,x) = 0$ will be given in terms of a generating differential operator L of the Markov process described by system (2.18) and having the form

$$LV = \frac{\partial V}{\partial t} + (\nabla V, a(t,x)) + \frac{1}{2}\sum_{r=1}^{k} (\nabla, b_r(t,x))^2 \, V, \tag{2.20}$$

where $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ and $(\,\cdot\,,\,\cdot\,)$ is the scalar product.

Theorem 2.3. *Let the above conditions be satisfied. If*

$$LV(t,x) \leq 0 \tag{2.21}$$

in the domain $\{t \geq 0\} \times D$*, then the set*

$$V(t,x) = 0, \ \ t \geq 0, \ \ x \in D \tag{2.22}$$

is positively invariant for (2.18)*. If, moreover,*

$$\inf_{t \geq 0, x \in D : \rho(N_t, x) > \delta} V(t,x) = V_\delta > 0 \tag{2.23}$$

for $\delta > 0$*, then set* (2.22) *is stochastically stable.*

Remark. If the function V, satisfying the conditions of the theorem, depends only on x, then it is clear that condition (2.23) is satisfied.

Proof. Without loss of generality, we will give a proof for $t_0 = 0$. Consider a solution $x(t,x_0)$ of (2.18) such that $x(0,x_0) = x_0 \in N_0$. Since N is closed in D, we see that N_0 is closed in D and, hence, $x(t,x_0) \in D$ for t in some interval $[0, \ \tau_D)$ by continuity of $x(t,x_0)$ with respect to t. Here τ_D is the time at which $x(t, x_0)$ leaves D for the first time. Clearly, $\tau_D > 0$ with probability 1. Denote $\tau_D(t) = \min\{\tau_D, t\}$. Then, applying the Ito formula to the process $V(t, x(t, x_0))$ and using a lemma in [70, p. 110] we get

$$\mathbf{E}\, V(\tau_D(t), x(\tau_D(t), x_0)) \; - V(0, x_0) = \mathbf{E} \int\limits_0^{\tau_D(t)} LV(s, x(s, x_0))ds.$$

In view of (2.21), this yields

$$\mathbf{E} V(\tau_D(t), x(\tau_D(t), x_0)) \le 0.$$

Since V is nonnegative in the domain $\{t > 0\} \times D$, we see that the inequality

$$V(\tau_D(t), x(\tau_D(t), x_0)) = 0 \tag{2.24}$$

holds with probability 1. This means that $(\tau_D(t), x(\tau_D(t), x_0)) \in N$ with probability 1. But since the projection of N onto $\mathbf{R}^n$ is closed in D, we have that $x(\tau_D(t), x_0)$ is an interior point of D with probability 1. This means that $\tau_D(t)$ is not the time of τ_D leaving the domain D. Hence, $\tau_D(t) = t$ with probability 1. It follows from (2.24) that $V(t, x(t, x_0)) = 0$ with probability 1 for all $t \ge 0$. This gives

$$\mathbf{P}\left\{ \sup_{t_i \in Q^+} V(t_i, x(t_i, x_0)) = 0 \right\} = 1\,,$$

where Q^+ is the set of nonnegative rational numbers. Using continuity we obtain that

$$\mathbf{P}\left\{ \sup_{t_i \in Q^+} V(t_i, x(t_i, x_0)) = 0 \right\} = \mathbf{P}\left\{ \sup_{t \ge 0} V(t, x(t, x_0)) = 0 \right\} = 1\,.$$

The latter identity yields

$$\mathbf{P}\{(t, x(t, x_0)) \in N, \ \forall t \ge 0\} = 1\,.$$

Let us now prove that the set N is stochastically stable. Let ε_1 and ε_2 be arbitrary positive constants such that the ε_1-neighborhood of the set N_t, together with its boundary, is contained in D for every $t \ge 0$. Since the projection of N is compact in D, this can always be done. Denote by V_{ε_1} the quantity

$$V_{\varepsilon_1} = \inf_{t \ge 0, x \in D: \rho(N_t, x) > \varepsilon_1} V(t, x)\,.$$

Conditions of the theorem imply that $V_{\varepsilon_1} > 0$. Since the process

$$V(\tau_{U_{\varepsilon_1}}(t), x(\tau_{U_{\varepsilon_1}}(t), t_0, x_0))$$

is a martingale, U_{ε_1} is an ε_1-neighborhood of N_t, similarly to a theorem in [70, p. 207] one can show that the solution $x(t, t_0, x_0)$ of system (2.18) satisfies the estimate

$$\mathbf{P}\left\{\sup_{t\geq t_0} \rho(x(t, t_0, x_0), N_t) > \varepsilon_1\right\} \leq \frac{V(t_0, x_0)}{V_{\varepsilon_1}}. \tag{2.25}$$

Choose now a δ-neighborhood of the set N_0 in such a way that

$$V(t_0, x_0) \leq \ V_{\varepsilon_1}\varepsilon_2 \,. \tag{2.26}$$

By [139, p. 61] such a choice can always be made. A use of (2.25) and (2.26) finishes the proof of the theorem. $\square$

Let us now consider locally invariant sets for system (2.18).

Let $S \in R^{n+1}$ be some set, closed and nonempty for every $t \geq 0$. Take an arbitrary $t_0 \geq 0$. Denote

$$\tau(t_0, x_0) = \inf\{t > t_0 : (t, x(t, t_0, x_0)) \notin S\}\,.$$

It is clear that $\tau(t_0, x_0)$ is a Markov moment with respect to the flow F_t, where F_t is a flow of σ-algebras in the definition of a solution of equation (2.18).

Definition 2.7. A set S is called *locally invariant* for system (2.18) if $(t_0, x_0) \in S$ implies that $\tau(t_0, x_0) > 0$ with probability 1.

Theorem 2.4. *Let there exist a nonnegative Lyapunov function $V(t, x)$ on the domain $\{t \geq 0\} \times \overline{D}$ such that $LV(t, x) \leq 0$. Then the set $V(t, x) = 0,\ x \in D$, if it is nonempty, is locally invariant for* (2.18).

Proof. Let $t_0 \geq 0,\ x_0 \in D$ be such that $V(t_0, x_0) = 0$. Then, since the solution $x(t, t_0, x_0)$ is continuous, we see that $\tau_U > 0$ with probability 1, where τ_U- is the time at which the solution $x(t, t_0, x_0)$ leaves some neighborhood U of the point $x_0 \in D$.

This means that (2.24) holds on the interval $[t_0,\ \tau_D)$, which finishes the proof of the theorem. $\square$

Let us now give an example that illustrates the results obtained in this section.

Example. Consider a system of stochastic Ito equations,

$$dx = \ -xdt \ -ydw(t), \ dy = \ -ydt \ +xdw(t)\,, \tag{2.27}$$

on the domain $x^2+y^2<2$, $t\geq 0$, where $w(t)$ is a Wiener process. Let us show that the set S of points satisfying the equation

$$x^2+y^2=\exp\{-t\}$$

for $t\geq 0$ is invariant for (2.27) and is stochastically stable.

Indeed, take the Lyapunov function mentioned in Theorem 2.3 to be $V = (x^2+y^2-\exp\{-t\})^2$.

The projection of the set N_0 is a set of points (x,y) in the disk $x^2+y^2\leq 1$, compact in the domain $x^2+y^2<2$. It is easy to see that

$$L\,V = \ 2(x^2+y^2-\exp\{-t\})\ (\exp\{-t\}-x^2-y^2) = -2V \leq 0.$$

Hence, inequality (2.21) is satisfied. It is immediate that the function V satisfies condition (2.23).

2.3 The behaviour of invariant sets under small perturbations

The questions of existence of invariant sets and their behaviour under small perturbations where considered in [139, p. 74] for differential systems of the form

$$\frac{dx}{dt} = X(x) + \mu Y(x)\,, \tag{2.28}$$

where μ is a small positive parameter, assuming that the unperturbed system ($\mu = 0$) has an asymptotically stable invariant set. It was found that system (2.28), if a small perturbation is present, also has an invariant set, although its topological structure can be significantly different from the invariant set for the unperturbed system.

The aim of this section is to study the case where the perturbation in (2.28) is random and the system itself is of the form

$$\frac{dx}{dt} = X(x) + \mu Y(x,\xi(t))\,, \tag{2.29}$$

where $\xi(t)$ is a stochastically continuous random process taking values in $\mathbf{R}^m$, $t\geq 0$.

Together with system (2.29), consider the unperturbed system

$$\frac{dx}{dt} = X(x) \tag{2.30}$$

and assume that it has a compact asymptotically stable, positively invariant set $M_0 \subset \mathbf{R}^n$. We will formulate conditions that would imply that system (2.29) also has a set that is positively invariant in the sense of Definition 2.2.

Theorem 2.5. *Let the functions $X(x)$ and $Y(x,y)$ in system* (2.29) *be defined and continuous in x that runs over a neighborhood M_0 and $y \in \mathbf{R}^m$. Also assume that they are Lipschitz continuous in x with a constant L.*

If there exists a positive constant C such that $|Y(x,y)| \le C$, then one can find $\mu_0 > 0$ such that, for arbitrary $\mu \le \mu_0$, system (2.29) *has a positively invariant set $M_\mu^t \subset \mathbf{R}^n$, $t \ge 0$, and*

$$\lim_{\mu\to 0} \sup_{t\ge 0} \rho(M_0, M_\mu^t) = 0\,. \tag{2.31}$$

Proof. Denote by $x(t,x_0)$ and $x(t,x_0,\mu)$ the solutions of systems (2.30) and (2.29), correspondingly, which take the value x_0 at $t = 0$. Then, for $t \in [0,\ T]$, we have

$$|x(t,x_0,\mu) - x(t,x_0)| \le L\int_0^t |x(s,x_0,\mu) - x(s,x_0)|\,ds + \mu CT.$$

Hence,

$$|x(t,x_0,\mu) - x(t,x_0)| \le \mu CT\exp\{LT\} \tag{2.32}$$

with probability 1 for arbitrary $t \in [0,\ T]$.

Since the set M_0 is asymptotically stable, we have

$$\lim_{t\to\infty} \rho(x(t,x_0), M_0) = 0$$

for $x_0 \in \overline{U}_\delta(M_0)$, a closed δ-neighborhood of the set M_0, and sufficiently small δ.

Fix $\delta_1 > 0$ and choose $\delta = \delta(\delta_1) > 0$ and $T = T(\delta) > 0$ such that

$$\rho(x(t,x_0), M_0) < \frac{\delta_1}{2},\ t \ge 0,\ \rho(x(t,x_0), M_0) < \frac{\delta}{2},\ t \ge T\,, \tag{2.33}$$

$x_0 \in \overline{U}_\delta(M_0)$.

For the chosen δ, δ_1, and T there is $\mu(\delta_1) > 0$, with $\lim_{\delta_1\to 0}\mu(\delta_1) = 0$ monotonically, such that

$$\rho(x(t,x_0,\mu), x(t,x_0)) < \frac{\delta}{2}$$

with probability 1 for $t \in [0,\ T]$ and all $x_0 \in \overline{U}_\delta(M_0)$ and $\mu \le \mu(\delta_1)$. These estimates yield

$$\rho(x(t, x_0, \mu), M_0) < \delta_1 \qquad \text{and} \qquad \rho(x(T, x_0, \mu), M_0) < \delta$$

with probability 1 for all $x_0 \in \overline{U}_\delta(M_0)$ if $\mu < \mu(\delta_1)$ and $t \in [0,\ T]$.

Note now that the above inequalities permit to replace the variable x_0 with $x_0(\omega)$, a random variable that belongs to $\overline{U}_\delta(M_0)$ with probability 1.

Consider a solution $x_1(t)$ of system (2.30) such that

$$x_1(T) = x(T, x_0, \mu)$$

for $t = T$. Then we have

$$|x(t, x_0, \mu) - x_1(t)| \le \mu C T \exp\{LT\} \tag{2.34}$$

on $[T, 2T]$ with probability 1. Since system (2.30) is autonomous and M_0 is compact, the system is asymptotically stable, hence $x_1(t)$ satisfies (2.32) with probability 1 for $\delta,\ T,\ t \ge T$ chosen as above.

Hence,

$$\rho(x(t, x_0, \mu), M_0) < \delta_1 \qquad \text{and} \qquad \rho(x(2T, x_0, \mu), M_0) < \delta$$

with probability 1 on the interval $[T,\ 2T]$ for $\mu < \mu(\delta_1)$.

Similar considerations show that solutions of system (2.29), which start in a δ-neighborhood of M_0 with probability 1, stay in the δ_1-neighborhood of this set with probability 1 for $t \ge 0$.

Consider now a set of curves $(t,\ x(t, x_0, \mu))$ in $\mathbf{R}^{n+1}$, where $x(t, x_0, \mu)$ are solutions of system (2.29), which belong to the δ_1-neighborhood of the set M_0 with probability 1 for $t \ge 0$. Let M_μ^t be the set of all such curves. Thus obtained set M_μ^t is clearly semi-invariant for system (2.29) and

$$M_\mu^t \subset \overline{U}_{\delta_1}(M_0)$$

for each fixed $t \ge 0$, implying that (2.31) holds, since δ_1 is arbitrary. $\square$

The main condition in the previous theorem that $|Y(x, y)| \le C$ is rather restrictive, which significantly limits applications of the theorem. It is natural to try to weaken the condition, which can be achieved by replacing it with the condition that the perturbing random process is bounded in probability. This requires a somewhat different definition of invariance of the set M_μ^t.

Definition 2.8. A set M_0 for system (2.30) is called *positively invariant in probability* for small perturbations if for arbitrary $\varepsilon > 0$ there exists $\mu_0 > 0$ such that for arbitrary μ, $0 < \mu \le \mu_0$, there is a set $M_\mu^t \subset \mathbf{R}^{n+1}$ satisfying

$$\mathbf{P}\{x(t, x_0, \mu) \in M_\mu^t,\ t \ge t_0\} > 1 - \varepsilon$$

with probability 1 for $x_0(\omega) \in M_\mu^{t_0}$.

Denote

$$\eta(t) = \sup_{x \in \mathbf{R}^n} |Y(x, \xi(t))|\,.$$

Theorem 2.6. *Let the functions $X(x)$, $Y(x,y)$ in system (2.29) be defined and continuous in $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, globally Lipschitz continuous with respect to x with a constant L.*

If $\eta(t)$ satisfies the condition

$$\mathbf{P}\left\{\sup_{k \in \mathbf{Z}^+} \int_{kT}^{(k+1)T} \eta(s)\, ds \ge r\right\} \to 0,\ r \to \infty\,, \tag{2.35}$$

for an arbitrary $T > 0$, then the set M_0 is positively invariant in probability in the sense of the definition above, and

$$\lim_{\mu \to 0} \sup_{t \ge 0} \rho(M_0, M_\mu^t) = 0\,. \tag{2.36}$$

Remark. Condition (2.35) is clearly satisfied if the process $\eta(t)$ is bounded in probability such that

$$\mathbf{P}\left\{\sup_{t \ge 0} \eta(t) > r\right\} \to 0,\ r \to \infty\,. \tag{2.37}$$

Proof. Let us first remark that the conditions of the theorem imply that solutions of system (2.29) are separable random processes.

Let us show that for arbitrary $\varepsilon > 0$, $\delta_1 > 0$ there exist $\delta = \delta(\delta_1)$, $T = T(\delta_1)$, $\mu_0 = \mu_0(\varepsilon, \delta_1)$ ($\lim_{\delta_1 \to 0} \mu_0(\varepsilon, \delta_1) = 0$ monotonically for arbitrary fixed $\varepsilon > 0$) such that

$$\mathbf{P}\{x(t, x_0, \mu) \in \overline{U}_{\delta_1}(M_0),\ t \ge 0\} > 1 - \varepsilon\,, \tag{2.38}$$

$$\mathbf{P}\{x(T, x_0, \mu) \in \overline{U}_{\delta}(M_0)\} > 1 - \varepsilon \tag{2.39}$$

for $\mu \le \mu_0(\varepsilon, \delta_1)$, where $x(t, x_0, \mu)$ is a solution of system (2.29) such that $x(0, x_0, \mu) = x_0(\omega)$ for all $x_0(\omega) \in \overline{U}_\delta(M_0)$ with probability 1.

Fix $\delta_1 > 0$ and $\varepsilon > 0$. Since M_0 is an asymptotically stable, compact, and invariant set for system (2.30), there exist $\delta = \delta(\delta_1),\ T = T(\delta)$ such that

$$\rho(x(t,x_0), M_0) < \frac{\delta_1}{2}, \quad t \geq 0, \quad \text{and} \quad \rho(x(t,x_0), M_0) < \frac{\delta}{2} \ \text{ for } t \geq T \tag{2.40}$$

with probability 1 for all $x_0(\omega) \in \overline{U}_\delta(M_0)$.

Let us estimate the difference between two solutions of systems (2.29) and (2.30) over $[0,\ T]$,

$$|x(t,x_0,\mu) - x(t,x_0)| \leq L \int_0^t |x(s,x_0,\mu) - x(s,x_0)|\, ds + \mu \int_0^T \eta(s)\, ds\,.$$

Hence,

$$|x(t,x_0,\mu) - x(t,x_0)| \leq \mu \exp\{LT\} \int_0^T \eta(s)\, ds \tag{2.41}$$

with probability 1.

It follows from condition (2.35) that for a given $\varepsilon > 0$ and the chosen T there exists $r_0(T)$ such that

$$\mathbf{P}\left\{ \sup_{k \in \mathbf{Z}^+} \int_{kT}^{(k+1)T} \eta(s)\, ds > r \right\} < \varepsilon$$

for arbitrary $r > r_0(T)$. And, hence,

$$\mathbf{P}\left\{ \int_0^T \eta(s)\, ds > r \right\} < \varepsilon\,.$$

Then

$$\begin{aligned} \mathbf{P}\left\{ |x(t,x_0,\mu) - x(t,x_0) > \frac{\delta}{2} \right\} &\leq \mathbf{P}\left\{ \mu \exp\{LT\} \int_0^T \eta(s)\, ds > \frac{\delta}{2} \right\} \\ &\leq \mathbf{P}\left\{ \sup_{k \in \mathbf{Z}^+} \int_{kT}^{(k+1)T} \eta(s)\, ds > \frac{\delta}{2\mu \exp\{LT\}} \right\} < \varepsilon \end{aligned} \tag{2.42}$$

for all $\mu \leq \mu_0(\varepsilon, \delta_1)$, where $\mu_0(\varepsilon, \delta_1)$ is such that

$$\frac{\delta}{2\mu_0 \exp\{LT\}} > r_0(T)\,.$$

Denote by A_{μ_0} the complement of the set

$$\left\{ \omega : \sup_{k \in \mathbf{Z}^+} \int_{kT}^{(k+1)T} \eta(s)\, ds > \frac{\delta}{2\mu_0 \exp\{LT\}} \right\}.$$

It is clear that

$$\mathbf{P}(A_{\mu_0}) > 1 - \varepsilon\,. \tag{2.43}$$

It follows from the above that

$$\begin{aligned}
&\mathbf{P}\{\rho(x(t, x_0, \mu), M_0) > \delta_1\} \\
&\quad \leq \mathbf{P}\{\rho((x(t, x_0, \mu), x(t, x_0)) + \rho(x(t, x_0), M_0) > \delta_1\} \\
&\quad \leq \mathbf{P}\left\{\rho(x(t, x_0), M_0) > \frac{\delta_1}{2}\right\} + \mathbf{P}\left\{\rho(x(t, x_0, \mu), x(t, x_0)) > \frac{\delta_1}{2}\right\} < \varepsilon
\end{aligned} \tag{2.44}$$

for $t \in [0,\ T]$ and that

$$\mathbf{P}\{\rho(x(T, x_0, \mu), M_0) \geq \delta\} < \varepsilon\,. \tag{2.45}$$

It is clear that, for arbitrary $t \in [0,\ T]$, the following holds:

$$\{\omega : \rho(x(t, x_0, \mu), M_0) > \delta_1\} \subset \overline{A}_{\mu_0}\,. \tag{2.46}$$

This yields

$$A_{\mu_0} \subset \overline{\{\omega : \rho(x(t, x_0, \mu), M_0) > \delta_1\}} \tag{2.47}$$

for all $t \in [0,\ T]$. Hence,

$$A_{\mu_0} \subset \bigcap_{t \in [0,\ T]} \{\omega : \rho(x(t, x_0, \mu), M_0) \leq \delta_1\}\,, \tag{2.48}$$

which gives

$$A_{\mu_0} \subset \left\{\sup_{t \in [0,\ T]} \rho(x(t, x_0, \mu), M_0) \leq \delta_1\right\}. \tag{2.49}$$

Note that the set in the right-hand side of (2.48) is measurable since $x(t, x_0, \mu)$ is separable. Using (2.43) we have

$$\mathbf{P}\left\{\sup_{t \in [0,\ T]} \rho(x(t, x_0, \mu), M_0) \leq \delta_1\right\} > 1 - \varepsilon \tag{2.50}$$

and

$$A_{\mu_0} \subset \{\omega : \rho(x(T, x_0, \mu), M_0) \leq \delta\} \tag{2.51}$$

for all $\mu \leq \mu_0(\varepsilon, \delta_1)$. Hence, solutions of (2.29) satisfying $x_0(\omega) \in \overline{U}_\delta(\mu_0)$ with probability 1 do not leave the δ_1-neighborhood of the set M_0 and $x(T, x_0, \mu) \in \overline{U}_\delta(M_0)$ with probability greater than $1-\varepsilon$. Hence, the trajectories of solutions satisfying $\omega \in A_{\mu_0}$ do not leave $\overline{U}_{\delta_1}(M_0)$.

Now, consider a solution of the unperturbed equation (2.28), $x_1(t, x_T)$, such that $x_1(T, x_T) = x_T$, where $x_T = x(T, x_0, \mu)$ on the interval $[T,\ 2T]$.

Since the system (2.28) is autonomous, the set M_0, being compact, is uniformly asymptotically stable, which means that estimates (2.40) do not depend on the initial conditions. Hence, inequalities (2.40) hold for all $\omega \in \Omega$ such that $x(T, x_0, \mu) = x_T(\omega) \in \overline{U}_\delta(M_0)$ and, in particular, for $\omega \in A_{\mu_0}$.

We have the estimate

$$|x(t, x_0, \mu) - x_1(t, x_T)| \le \int_T^t |x(s, x_0, \mu) - x_1(s, x_T)|\, ds + \mu \int_T^{2T} \eta(s)\, ds$$

on $[T,\ 2T]$, which implies that

$$|x(t, x_0, \mu) - x_1(t, x_T)| \le \mu \exp\{LT\} \int_T^{2T} \eta(s)\, ds\,. \tag{2.52}$$

Since

$$\left\{\omega : \int_T^{2T} \eta(s)\, ds > \frac{\delta}{2\mu \exp\{LT\}}\right\} \subset \overline{A}_{\mu_0}$$

for all $\omega \in A_{\mu_0}$, we have on $[T,\ 2T]$, as before, that $x(t, x_0, \mu) \in \overline{U}_{\delta_1}(M_0)$ and $x(2T, x_0, \mu) \in \overline{U}_\delta(M_0)$ for $\mu \le \mu_0(\varepsilon, \delta_1)$. Similar considerations yield that $x(t, x_0, \mu) \in \overline{U}_{\delta_1}(M_0)$ for arbitrary $k \in \mathbf{Z}^+$ and $\omega \in A_{\mu_0}$ if $t \in [kT,\ (k+1)T]$, and $x(t, x_0, \mu) \in \overline{U}_\delta(M_0)$ at the endpoints of the interval for $\mu \le \mu_0(\varepsilon,\ \delta_1)$, which proves inequalities (2.38) and (2.39).

Let the set M_μ^t in R^{n+1} be composed of the curves $(t, x(t, x_0, \mu, \omega))$, where $x(t, x_0, \mu, \omega)$ is a trajectory of system (2.29) for $\omega \in A_{\mu_0}$.

It is clear that the inequality in the above definition is true for such a set. The limit relations (2.36) can be proved similarly to Theorem 2.5. □

Remark 2. A stability problem for the zero solution of the system

$$\frac{dx}{dt} = F(t, x) \tag{2.53}$$

with permanently acting random perturbations was studied in [70, Ch. 1, Sect. 6]. There were found conditions such that a solution of the system

$$\frac{dx}{dt} = F(t, x) + \mu G(t, x, \xi(t))$$

that starts in a sufficiently small neighborhood of the origin would not leave a given neighborhood of zero with a sufficiently large probability if the acting perturbations are small in the mean. It was shown there that a sufficient condition is exponential stability of the zero solution of system (2.53). It follows from the proof of the preceding theorem that if the random perturbations satisfy condition (2.35), then the condition of exponential stability of the zero solution can be weakened and replaced with the condition of asymptotic stability of the zero solution, uniform in $t_0 \geq 0$.

Let us now consider similar questions in the case where the perturbed system (2.29) is a system of stochastic Ito equations of the form

$$dx = X(x)dt + \mu Y(t,x)dW(t)\,, \tag{2.54}$$

where $W(t)$ is a many dimensional Wiener process with independent components and $Y(t,x)$ is a matrix of the corresponding dimensions. Let the coefficients of system (2.54) satisfy the conditions for existence and strong uniqueness of the Cauchy problem for $t \geq 0$. For such systems, condition (2.35) can be written in a simpler form.

Denote

$$u(t) = \sup_{x \in \mathbf{R}^n} |Y(t,x)|.$$

Theorem 2.7. *If*

$$\int_0^\infty u^2(t)\,dt < \infty\,, \tag{2.55}$$

then the statements of Theorem 2.6 hold true for the set M_0 of system (2.30).

Proof. Again, denote by $x(t,x_0,\mu)$ a solution of system (2.54) and by $x(t,x_0)$ a solution of system (2.30). Then, on the interval $[0,\ T]$, we have

$$\begin{aligned}|x(t,x_0,\mu) - x(t,x_0)| \leq \int_0^t & L|x(s,x_0,\mu) - x(s,x_0)|\,ds \\ &+ \mu \sup_{t \in [0,\ T]} \left| \int_0^t Y(s, x(s,x_0,\mu))\,dW(s) \right|,\end{aligned}$$

which gives

$$|x(t,x_0,\mu) - x(t,x_0)| \leq \exp\{LT\}\mu \sup_{t \in [0,\ T]} \left| \int_0^t Y(s, x(s,x_0,\mu))\,dW(s) \right|. \tag{2.56}$$

Hence, to finish the proof, it is sufficient to show that

$$\mathbf{P}\left\{ \sup_{k \in \mathbf{Z}^+} \sup_{t \in [kT,\ (k+1)T]} \left| \int_{kT}^t Y(s, x(s,x_0,\mu))\,dW(s) \right| > r \right\} \to 0$$

as $r \to \infty$. We have

$$\begin{aligned}
&\mathbf{P}\left\{\sup_{k\in\mathbf{Z}^+}\ \sup_{t\in[kT,\ (k+1)T]}\left|\int_{kT}^{t} Y(s,x(s,x_0,\mu))\,dW(s)\right| > r\right\} \\
&\le \sum_{k=0}^{\infty}\mathbf{P}\left\{\sup_{t\in[kT,\ (k+1)T]}\left|\int_{kT}^{t} Y(s,x(s,x_0,\mu))\,dW(s)\right| > r\right\} \\
&\le \sum_{k=0}^{\infty}\frac{n^2}{r^2}\int_{kT}^{(k+1)T}\mathbf{E}|Y(s,x(s,x_0,\mu))|^2\,ds \le \sum_{k=0}^{\infty}\frac{n^2}{r^2}\int_{kT}^{(k+1)T} u^2(t)\,dt \\
&= \frac{n^2}{r^2}\int_0^{\infty} u^2(t)\,dt \to 0,\ r\to\infty\,.
\end{aligned}$$

The rest of the argument repeats that of the proof of Theorem 2.6. □

2.4 A study of stability of an equilibrium via the reduction principle for systems with regular random perturbations

Stability of the zero solution of the system

$$\frac{dx}{dt} = Ax + X(x,y),\ \frac{dy}{dt} = By + Y(x,y)\,, \tag{2.57}$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, was studied in [123] if all eigen values of the matrix A have zero real parts, and all eigen values of the matrix B have negative real parts. It was shown there that, if the functions X, Y are Lipschitz continuous in a neighborhood of zero with a sufficiently small constant, then system (2.57) has an invariant manifold, $y = f(x)$, and stability of the zero solution is equivalent to its stability on this manifold,

$$\frac{dx}{dt} = \ Ax + \ X(x, f(x))\,, \tag{2.58}$$

which is called in the theory of stability the *reduction principle*; some of its ideas can be traced back to works of Poincare.

In this section we obtain similar results for systems with random right-hand side under the assumption that the invariant manifold is the hyperplane $y = 0$.

Consider a system of ordinary differential equations, perturbed with a random process $\xi(t)$ having continuous trajectories, defined on a probability space

(Ω, F, P), and taking values in $\mathbf{R}^k$,

$$\frac{dx}{dt} = X(x,y),\ \frac{dy}{dt} = A(t)y + Y(t,x,y,\xi(t))\,. \tag{2.59}$$

We will assume that the functions $X(x,y)$ and $Y(t,x,y,z)$ are jointly continuous with respect to their variables on the domain $\{t \geq 0\} \times D_x \times D_y \times \mathbf{R}^k$, where D_x and D_y are some domains in $\mathbf{R}^n$ and $\mathbf{R}^m$, correspondingly (D_x contains the point $x = 0$ and D_y the point $y = 0$), $X(0,0) = 0$, $Y(t,x,0,z) \equiv 0$, and are Lipschitz continuous with respect to the variables x, y on this domain, that is,

$$\begin{aligned} |X(x_1,y_1) - X(x_2,y_2)| &\leq L_1\ (|x_1 - x_2| + |y_1 - y_2|), \\ |Y(t,x_1,y_1,z) - Y(t,x_2,y_2,z)| &\leq L_2\ (|x_1 - x_2| + |y_1 - y_2|). \end{aligned} \tag{2.60}$$

With these conditions, system (2.59) has the invariant manifold $y = 0$ on which it takes the form

$$\frac{dx}{dt} = X(x,0), \tag{2.61}$$

that is, it is deterministic.

For the matrix $A(t)$, we will assume that the fundamental matrix $\Phi(t,\tau)$ of the linear system

$$\frac{dy}{dt} = A(t)y$$

admits the estimate

$$\|\Phi(t,\tau)\| \leq R\exp\{-\rho(t-\tau)\} \tag{2.62}$$

with positive constants R and ρ independent of t and τ.

Let the zero solution of system (2.61) be asymptotically stable. We will show that the zero solution of system (2.59) with random perturbations is stable with probability 1.

Theorem 2.8. *Let $L_2 < \frac{\rho}{R}$ and the above conditions be satisfied. If the zero solution of system* (2.61) *is asymptotically stable, then the zero solution of system* (2.59) *is stable with probability* 1*, uniformly in t_0.*

Proof. Without loss of generality, the theorem can be proved for $\tau \geq 0$, as follows from estimate (2.62) uniform in $t_0 = 0$.

Let $x = x(t,x_0,y_0)$, $y = y(t,x_0,y_0)$ be a solution of system (2.59) and $x(0,x_0,y_0) = x_0$, $y(0,x_0,y_0) = y_0$. Let us show that for an arbitrary $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $|x_0| + |y_0| < \delta_1$ implies that $|x(t,x_0,y_0)| < \varepsilon$ and $|y(t,x_0,y_0)| < \varepsilon$ for $t \geq 0$ with probability 1.

It follows from a theorem in [70, p. 26] that system (2.59) has a solution and it is strongly unique for $t \geq 0$ until it leaves the domain where the right-hand side of the system is defined.

Denote $\tau_{D_{x,y}}(t) = \min\{t, \tau_{D_x}, \tau_{D_y}\}$, where τ_{D_x} is the time at which $x(t, x_0, y_0)$ enters the boundary of D_x, and by τ_{D_y} the time at which $y(t, x_0, y_0)$ enters the boundary of D_y. Differentiation shows that the second equation in (2.59) is equivalent, for $t < \tau_{D_{x,y}}$, to the integral equation

$$y(t, x_0, y_0) = \Phi(t, 0)y_0 + \int_0^t \Phi(t, \tau)Y(\tau, x(\tau, x_0, y_0), y(\tau, x_0, y_0), \xi(\tau))d\tau.$$

Using (2.60) and (2.62), and since $Y(t, x, 0, z) \equiv 0$, we have the estimate

$$\begin{aligned} |y(t, x_0, y_0)| &= |\Phi(t, 0)y_0| \\ &\quad + \int_0^t R\exp\{-\rho(t-\tau)\}|Y(\tau, x(\tau, x_0, y_0), y(\tau, x_0, y_0), \xi(\tau))|d\tau \\ &\leq R\exp\{-\rho t\}|y_0| + \int_0^t R\exp\{-\rho(t-\tau)\}L_2|y(\tau, x_0, y_0)|d\tau\,. \end{aligned}$$

This estimate is equivalent to the inequality

$$\exp\{\rho t\}|y(t, x_0, y_0)| \leq R|y_0| + RL_2\int_0^t \exp\{\rho\tau\}|y(\tau, x_0, y_0)|d\tau\,. \tag{2.63}$$

Using the Gronwall–Bellman inequality we have

$$\exp\{\rho t\}|y(t, x_0, y_0)| \leq R|y_0|\exp\{RL_2 t\},$$

which is equivalent to the estimate

$$|y(t, x_0, y_0)| \leq R|y_0|\exp\{-(\rho - RL_2)t\}. \tag{2.64}$$

Let us show that, if y_0 has a sufficiently small norm, then

$$\tau_{D_{x,y}}(t) = \tau_{D_x}(t) = \min\{t, \tau_{D_x}\}$$

with probability 1. Indeed, let $|y_0| < \delta$, where $\delta > 0$. If $\tau_{D_y} < \min\{t, \tau_{D_x}\}$ with positive probability, then there exists a set $A \in \Omega$ such that $P(A) > 0$, and for arbitrary $\omega \in A$ the solution $y(t, x_0, y_0, \omega)$ enters the boundary of the domain D_y in time less than $\tau_{D_x}(t)$. Then there exists $T(\omega) < \tau_{D_x}(t)$ such that $|y(T(\omega), x_0, y_0)| = \varepsilon$. Let δ be chosen so that $R|y_0| < \varepsilon$. It follows from inequality (2.64) that

$$\varepsilon = |y(T(\omega), x_0, y_0,)| \leq R|y_0| \exp\{-(\rho - RL_2)T(\omega)\} < \varepsilon,$$

which leads to a contradiction.

Consider now $x(t, x_0, y_0)$, a solution of the first equation. Denote by $\overline{x}(t, x_0)$ a solution of system (2.61). Since the zero solution is asymptotically stable, there is $\delta < \varepsilon$ such that $\overline{x}(t, x_0) \to \infty$ for $t \to 0$ uniformly in $|x_0| \leq \delta$. We will assume that $|\overline{x}(t, x_0)| < \frac{\varepsilon}{2}$ for $t \geq 0$.

Take $T > 0$ such that $|\overline{x}(t, x_0)| < \frac{\delta}{2}$ for $t \geq T$. Estimate the difference $x(t, x_0, y_0) - \overline{x}(t, x_0)$ on the interval $[0,\ T]$ until the solution enters the boundary of D_x.

We have

$$\begin{aligned}
|x(t, x_0, y_0) - \overline{x}(t, x_0)| &= \left| \int_0^t [X(x(s, x_0, y_0), y(s, x_0, y_0)) - X(\overline{x}(t, x_0), 0)] ds \right| \\
&\leq L_1 \left(\int_0^t |x(s, x_0, y_0) - \overline{x}(s, x_0)| ds + \int_0^t |y(s, x_0, y_0)| ds \right) \\
&\leq RL_1 |y_0| \int_0^t \exp\{-(\rho - RL_2)s\} ds \\
&\quad + L_1 \int_0^t |x(s, x_0, y_0) - \overline{x}(s, x_0)| ds \\
&\leq \frac{RL_1}{\rho - RL_2} |y_0| + L_1 \int_0^t |x(s, x_0, y_0) - \overline{x}(s, x_0)| ds \,.
\end{aligned}$$

Using the Gronwall-Bellman inequality, we obtain that

$$|x(t, x_0, y_0) - \overline{x}(t, x_0)| \leq \frac{RL_1}{\rho - RL_2} |y_0| \exp\{L_1 T\} \,.$$

Hence,

$$\begin{aligned}
|x(t, x_0, y_0)| &\leq |\overline{x}(t, x_0)| + |x(t, x_0, y_0) - \overline{x}(t, x_0)| \\
&\leq \frac{\varepsilon}{2} + \frac{RL_1}{\rho - RL_2} |y_0| \exp\{L_1 T\} \,. \qquad (2.65)
\end{aligned}$$

Choose $|y_0|$ so small that

$$\frac{RL_1}{\rho - RL_2}|y_0| \exp\{L_1 T\} < \frac{\delta}{2}\,. \tag{2.66}$$

It follows from (2.65) and (2.66) that $|x(t, x_0, y_0)| \le \varepsilon$ holds for $t \le T$ until the solution leaves the domain. Since trajectories of the process $x(t, x_0, y_0)$ are continuous, it follows that $x(t, x_0, y_0) \in D_x$ with probability 1 for $t \in [0,\ T]$.

It also follows from (2.65) and (2.66) that

$$|x(T, x_0, y_0)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta\,. \tag{2.67}$$

Consider the behavior of the solution $x(t, x_0, y_0)$ on the interval $[T,\ 2T]$. Denote by $\overline{x}_T(t)$ the solution of system (2.61) such that $\overline{x}_T(T) = x(T, x_0, y_0)$. Since $|\overline{x}_T(T)| < \delta$ and system (2.61) is asymptotically stable, it follows that $|\overline{x}_T(t)| < \frac{\varepsilon}{2}$ for $t \ge T$, and $|\overline{x}_T(t)| < \delta$ for $t \ge 2T$.

Hence, similarly to the above, we have on the interval $[T,\ 2T]$ that

$$|x(t, x_0, y_0) - \overline{x}_T(t)| \le RL_1|y_0|\frac{\exp\{-(\rho - RL_2)T\}}{\rho - RL_2} + L_1 \int\limits_T^t |x(s, x_0, y_0) - \overline{x}_T(s)|ds,$$

whence it follows that

$$|x(t, x_0, y_0) - \overline{x}_T(t)| \le RL_1|y_0|\frac{\exp\{-(\rho - RL_2)T\}}{\rho - RL_2}\exp\{L_1 T\}.$$

This inequality and (2.66) yield the estimate

$$|x(t, x_0, y_0)| \le \frac{\delta}{2} + |\overline{x}_T(t)| < \varepsilon,$$

which holds for $t \in [T,\ 2T]$ until the time at which the solution leaves the domain D_x. However, as before, the solution $x(t, x_0, y_0)$ does not leave the domain D_x for $t \in [T,\ 2T]$. For $t = 2T$, we have $|x(2T, x_0, y_0)| < \delta$.

Continuing this procedure we see that the solution $x(t, x_0, y_0)$ does not leave the ε-neighborhood of zero with probability 1. This and inequality (2.64) proves the theorem. $\square$

2.5 Stability of an equilibrium and the reduction principle for Ito type systems

We now consider systems of Ito type stochastic differential equations.

For $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, $t \geq 0$, consider an Ito type system of the form

$$dx = X(x,y)dt, \ dy = A(t)ydt + \sigma(t,x,y)dW(t). \tag{2.68}$$

Here $\sigma(t,x,y)$ is an $m \times r$-dimensional matrix, $W(t)$ an r-dimensional Wiener processes with independent components.

Let $X(x,y)$ and $\sigma(t,x,y)$ be Lipschitz continuous with respect to x, y for $t \geq 0$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, with constants L_1 and L_2, correspondingly. Moreover, we will assume that the coefficients of system (2.68) satisfy conditions that imply existence and strong uniqueness of a solution of the Cauchy problem for $t \geq 0$. Let the matriciant of the linear system with the matrix $A(t)$ satisfy condition (2.62).

Let $y = 0$ be an invariant set for system (2.68), hence, $\sigma(t,x,0) \equiv 0$. In the entire subsection, we will always assume that system (2.68) satisfies the above conditions.

We will study *mean square stability* of the zero solution of system (2.68). Recall that, according to [186, p. 123], this means that for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|z(t,t_0,z_0)\|_2 = (\mathbf{E}|z(t,t_0,z_0)|^2)^{\frac{1}{2}} < \varepsilon \tag{2.69}$$

for $t \geq t_0$ and $\|z_0\|_2 < \delta$, where $z = (x,y)$ and $z_0 = (x_0, y_0)$, x_0 and y_0 are random variables independent of $w_r(t) - w_r(t_0)$, $r = \overline{1,k}$.

We will show that conditions for mean square stability of the zero solution of system (2.68) can be obtained from stability of the deterministic system (2.61) with random initial conditions.

We will say that the zero solution of system (2.61) is *mean square asymptotically stable uniformly in* $x_0(\omega)$ if there exists $\delta_1 > 0$ such that the limit relation

$$\lim_{t\to\infty} \|x(t,t_0,x_0(\omega)\|_2 = 0$$

holds uniformly in $x_0(\omega)$, where $x_0(\omega)$ is a random variable satisfying the condition $\|x_0(\omega)\|_2 < \delta_1$.

Theorem 2.9. *Let the zero solution of system* (2.61) *be mean square asymptotically uniformly stable and* $L_2 < \frac{(2\rho)^{\frac{1}{2}}}{R}$. *Then the zero solution of system* (2.68) *is also mean square stable uniformly in* $t_0 \geq 0$.

Proof. Clearly, it is sufficient to prove the theorem for $t_0 = 0$. Let $x = x(t,x_0,y_0)$, $y = y(t,x_0,y_0)$ be a solution of system (2.68) such that $x(0,x_0,y_0) = x_0$, $y_0(0,x_0,y_0) = y_0$, x_0 and y_0 are random processes independent of $w_r(t)$.

The proof of the theorem significantly uses an analogue of the Cauchy formula for stochastic equations obtained in [186, p. 230].

Let us show that the solution $y(t, x_0, y_0)$ admits the following representation:

$$y(t, x_0, y_0) = \Phi(t, 0)y_0 + \int_0^t \Phi(t, \tau)\sigma(\tau, x(\tau, x_0, y_0), y(\tau, x_0, y_0))dW(\tau). \quad (2.70)$$

Indeed, it follows from [186, p. 264] that the random process

$$\eta(t) = \int_0^t \Phi(t, \tau)\sigma(\tau, x(\tau, x_0, y_0), y(\tau, x_0, y_0))dW(\tau)$$

has the stochastic differential

$$\begin{aligned} d\eta(t) = &\left(\int_0^t \frac{\partial}{\partial t}\Phi(t, \tau)\sigma(\tau, x(\tau, x_0, y_0), y(\tau, x_0, y_0))dW(\tau) \right) dt \\ &+ \Phi(t, t)\sigma(t, x(t, x_0, y_0), y(t, x_0, y_0))dW(t) \end{aligned} \quad (2.71)$$

Using now properties of the fundamental matrix, we have

$$\begin{aligned} dy &= \frac{d}{dt}\Phi(t, 0)y_0 dt + d\eta(t) \\ &= A(t)\Phi(t, 0)y_0 dt + \left(\int_0^t A(t)\Phi(t, \tau)\sigma(\tau, x(\tau, x_0, y_0), y(\tau, x_0, y_0))dW(\tau) \right) dt \\ &\quad + \sigma(t, x(t, x_0, y_0), y(t, x_0, y_0))dW(t) = A(t)\Phi(t, 0)y_0 dt \\ &\quad + A(t)\left(\int_0^t \Phi(t, \tau)\sigma(\tau, x(\tau, x_0, y_0), y(\tau, x_0, y_0))dW(\tau) \right) dt \\ &\quad + \sigma(t, x(t, x_0, y_0), y(t, x_0, y_0))dW(t)\,, \end{aligned}$$

which proves (2.70).

Let us estimate the mean square norm of $y(t, x_0, y_0)$ using formula (2.70). We have

$$|y(t,x_0,y_0)|^2 \le 2(\|\Phi(t,0)\|^2 |y_0|^2 + \left| \int_0^t \Phi(t,\tau)\sigma(\tau, x(\tau,x_0,y_0), y(\tau,x_0,y_0))dW(\tau) \right|^2),$$

where $\|\cdot\|$ is the matrix norm.

It follows from properties of a stochastic integral that

$$\mathbf{E}|y(t,x_0,y_0)|^2 \le 2[\|\Phi(t,0)\|^2 \mathbf{E}|y_0|^2 + \int_0^t \|\Phi(t,\tau)\|^2 \mathbf{E}|\sigma(\tau, x(\tau,x_0,y_0), y(\tau,x_0,y_0))))|^2 d\tau].$$

Using (2.62) and the Lipschitz condition we get the estimate

$$\mathbf{E}|y(t,x_0,y_0)|^2 \le 2[R^2 \exp\{-2\rho t\} \mathbf{E}|y_0|^2 + \int_0^t R^2 \exp\{-2\rho(t-\tau)\} L_2^2 \mathbf{E}|y(\tau,x_0,y_0)|^2 d\tau],$$

which, together with the Gronwall-Bellman lemma yields the inequality

$$\mathbf{E}|y(t,x_0,y_0)|^2 \le 2R^2 \exp\{-(2\rho - (RL_2)^2)t\} \mathbf{E}|y_0|^2,$$

that is,

$$\|y(t,x_0,y_0)\|_2 \le \sqrt{2} R \exp\left\{-\left(\rho - \frac{1}{2}(RL_2)^2\right)t\right\} \|y_0\|_2. \tag{2.72}$$

Consider now a solution of the first equation. Denote by $\overline{x}(t, x_0)$ the solution of system (2.61) such that $\overline{x}(0, x_0) = x_0$. Take an arbitrary $\varepsilon > 0$. It follows from the conditions of the theorem that there exist $\delta < \varepsilon$ and $T > 0$ such that if $|x_0| < \delta$, then

$$\|\overline{x}(t,x_0)\|_2 < \frac{\varepsilon}{2}$$

for $t \ge 0$ and

$$\|\overline{x}(t,x_0)\|_2 < \frac{\delta}{2} \tag{2.73}$$

for $t \geq T$. Then, on the interval $[0,\ T]$, arguing as in the proof of Theorem 2.8 we have

$$|x(t,x_0,y_0)-\overline{x}(t,x_0)| \leq L_1\left(\int_0^t |x(s,x_0,y_0)-\overline{x}(s,x_0)|ds+\int_0^t |y(s,x_0,y_0)|ds\right).$$

Taking square of both sides of this inequality and using the Cauchy–Bunyakovskii inequality we get

$$\mathbf{E}|x(t,x_0,y_0)-\overline{x}(t,x_0)|^2 \leq 2\left(L_1^2 T\int_0^t \mathbf{E}|x(s,x_0,y_0)-\overline{x}(s,x_0)|^2 ds + T\int_0^t \mathbf{E}|y(s,x_0,y_0)|^2 ds\right),$$

which, with a use of (2.72), gives

$$\mathbf{E}|x(t,x_0,y_0)-\overline{x}(t,x_0)|^2 \leq \frac{2TR^2}{2\rho-(RL_2)^2}\mathbf{E}|y_0|^2 + 2L_1^2 T\int_0^t \mathbf{E}|x(s,x_0,y_0)-\overline{x}(s,x_0)|^2 ds\,.$$

Applying the Gronwall–Bellman lemma to the above inequality we get the estimate

$$\mathbf{E}|x(t,x_0,y_0)-\overline{x}(t,x_0)|^2 \leq \frac{2TR^2}{2\rho-(RL_2)^2}\exp\{2L_1^2T^2\}\mathbf{E}|y_0|^2\,, \tag{2.74}$$

which, finally, gives

$$\|x(t,x_0,y_0)-\overline{x}(t,x_0)\|_2 \leq \frac{R(2T)^{\frac{1}{2}}}{(2\rho-(RL_2)^2)^{\frac{1}{2}}}\exp\{L_1^2T^2\}\|y_0\|_2\,, \tag{2.75}$$

valid for $t \in [0,\ T]$.

Let $A = \frac{R(2T)^{\frac{1}{2}}}{(2\rho-(RL_2)^2)^{\frac{1}{2}}}\exp\{L_1^2T^2\}$. By choosing a sufficiently small δ_1, we can get that

$$A\|y_0\|_2 < \frac{\delta}{2}$$

for $\|y_0\|_2 < \delta_1$, hence, obtaining

$$\begin{aligned}\|x(t,x_0,y_0)\| &\leq \|\overline{x}(t,x_0)\|_2 + \|x(t,x_0,y_0) - \overline{x}(t,x_0)\|_2 \\ &< \frac{\varepsilon}{2} + \frac{\delta}{2} < \varepsilon \end{aligned} \tag{2.76}$$

that holds for $t \in [0,\ T]$. The latter inequality, with a use of (2.73), also yields that $\|x(T,x_0,y_0)\|_2 < \delta$.

Consider now the behaviour of the solution $x(t,x_0,y_0)$ on the interval $[T,\ 2T]$. To this end, denote by $\overline{x}_T(t)$ a solution of system (2.61) such that $\overline{x}_T(T) = x(T,x_0,y_0)$. Since $y(t,x_0,y_0)$ is a measurable process with respect to the standard σ-algebra F_t by the definition of a solution of system (2.68), and $x(t,x_0,y_0)$, being measurable with respect to the minimal σ-algebra generated by events of the form $\{x_0 \in B\}$, $\{y_0 \in C\}$, $\{y(s,x_0,y_0) \in D\}, s \in [0,\ t]$, where B, D, C are sets in the Borel σ-algebras on $\mathbf{R}^n, \mathbf{R}^m$, correspondingly, is also measurable with respect to F_t, we see that $x(T,x_0,y_0)$ does not depend on $w_r(t) - w_r(T)$, $r = \overline{1,k}$, where x_0 and y_0 are F_0-measurable. Note that $\|\overline{x}_T(T)\|_2 < \delta$.

A similar reasoning gives (2.76) on $[T,\ 2T]$ and the inequality

$$\|x(2T,x_0,y_0)\|_2 < \delta.$$

Similarly, we get

$$\|x(t,x_0,y_0)\|_2 < \varepsilon \tag{2.77}$$

for $t \geq 0$. Choosing $\delta_2 \leq \min\{\delta,\ \delta_1\}$ such that $\sqrt{2}R\delta_2 < \varepsilon$ we get the inequalities $\|x(t,x_0,y_0)\|_2 < \varepsilon$ and $\|y(t,x_0,y_0)\| < \varepsilon$ that hold for $t \geq 0$, $\|x_0\|_2 < \delta_2$, $\|y_0\| < \delta_2$, which proves the theorem. □

Let us remark that a similar statement for the mean square exponential stability can be obtained by using a result from [186, p. 131] assuming that the unperturbed system is mean square exponentially stable and the Lipschitz constants L_1 and L_2 are sufficiently small. Since Theorem 2.9, as opposed to the mentioned result, gives only mean square stability, it is natural that the generating system is assumed to satisfy a condition weaker than exponential stability, and the assumption that L_1 is small is not essential.

Theorem 2.9 permits to reduce a study of stability of the stochastic system (2.68) to a study of stability of a deterministic system with random initial conditions.

If we assume in the definition of mean square stability that the initial conditions are not random, then a study of the mean square stability of system (2.68) can be reduced to the question of stability of a deterministic system with initial conditions that are not random.

Theorem 2.10. *If system* (2.61) *in* $\mathbf{R}^n$ *has a positive definite quadratic form,* $V(x) = (Sx, x)$, *such that the derivative along* (2.61) *satisfies* $\frac{dV}{dt} = \dot{V} \leq 0$ *for* $x \in \mathbf{R}^n$, *then the zero solution of system* (2.68) *is mean square stable uniformly in* $t_0 \geq 0$.

Proof. Since the quadratic form $V(x)$ is positive definite, there exist numbers $\alpha > 0$ and $\beta > 0$ such that

$$\alpha|x|^2 \leq V(x) \leq \beta|x|^2. \tag{2.78}$$

Define a norm using $V(x)$ by setting

$$\|x(\omega)\|_V = (\mathbf{E}V(x(\omega)))^{\frac{1}{2}}. \tag{2.79}$$

By (2.78), the norms $\|\cdot\|_2$ and $\|\cdot\|_V$ are equivalent. For an arbitrary $\varepsilon > 0$, choose $\delta > 0$ such that $V(x_0) < (\frac{\varepsilon}{4})^2$ for $|x_0| < \delta$. Such a choice is possible, since $V(x)$ is continuous. Conditions of the theorem imply that the function $V(\overline{x}(t, x_0))$ is not increasing and, hence, $V(\overline{x}(t, x_0)) \leq V(x_0) \leq (\frac{\varepsilon}{4})^2$ for $t \geq 0$.

Fix a number $T > 0$ such that

$$\exp\{(-2\rho - (RL_2)^2)T\} < \frac{1}{4}. \tag{2.80}$$

It is easy to see that (2.74) holds on the interval $[0,\ T]$. Choose $\delta_1 > 0$ such that

$$A^2|y_0|^2 < \left(\frac{\varepsilon}{4}\right)^2 \frac{1}{\beta} \tag{2.81}$$

for $|y_0| < \delta_1$, where A is defined in Theorem 2.8.

Hence, we have

$$\mathbf{E}|x(t, x_0, y_0) - \overline{x}(t, x_0)|^2 \leq \left(\frac{\varepsilon}{4}\right)^2 \frac{1}{\beta}$$

for $t \in [0,\ T]$. This inequality leads to the estimate

$$\begin{aligned}\|x(t, x_0, y_0)\|_V &\leq \|x(t, x_0, y_0) - \overline{x}(t, x_0)\|_V + \|\overline{x}(t, x_0)\|_V \\ &\leq \sqrt{\beta \mathbf{E}|x(t, x_0, y_0) - \overline{x}(t, x_0)|^2} + \sqrt{V(\overline{x}(t, x_0))} < \frac{\varepsilon}{2}\end{aligned} \tag{2.82}$$

that holds for $t \in [0,\ T]$.

Consider the behavior of the solution $x(t, x_0, y_0)$ on $[T,\ 2T]$. As above, making an estimate of the difference $|x(t, x_0, y_0) - \overline{x}_T(t)|$, where $\overline{x}_T(t)$ is a solution of system (2.61) such that $\overline{x}_T(T) = x(T, x_0, y_0)$, we get for $t \in [T,\ 2T]$ that

$$\begin{aligned}\mathbf{E}|x(t,x_0,y_0) - \overline{x}_T(t)|^2 &\le 2L_1^2 T \int_T^t \mathbf{E}|x(s,x_0,y_0) - \overline{x}_T(s)|^2 ds \\ &\qquad + 2TR^2|y_0|^2 \int_T^t \exp\{-(2\rho - (RL_2)^2)s\} ds \\ &\le 2L_1^2 T \int_T^t \mathbf{E}|x(s,x_0,y_0) - \overline{x}_T(s)|^2 ds \\ &\qquad + 2TR^2|y_0|^2 \frac{\exp\{-(2\rho - (RL_2)^2)T\}}{2\rho - (RL^2)^2}\,.\end{aligned}$$

Using (2.80) and (2.81) we now get

$$\begin{aligned}\mathbf{E}|x(t,x_0,y_0) - \overline{x}_T(t)|^2 &\le 2TR^2|y_0|^2 \frac{\exp\{-(2\rho - (RL_2)^2)T\}}{2\rho - (RL^2)^2} \exp\{2L_1^2T^2\} \\ &< \left(\frac{\varepsilon}{4}\right)^2 \frac{1}{\beta}\frac{1}{4}\,. \end{aligned}\tag{2.83}$$

Hence, for $t \in [T,\ 2T]$, we obtain the inequality

$$\begin{aligned}\|x(t,x_0,y_0)\|_V &\le \sqrt{\beta \mathbf{E}|x(t,x_0,y_0) - \overline{x}_T(t)|^2} + \sqrt{\mathbf{E}V(\overline{x}_T(t))} \\ &< \frac{\varepsilon}{4}\frac{1}{2} + \sqrt{\mathbf{E}V(\overline{x}_(T,x_0))} \le \frac{\varepsilon}{4}\frac{1}{2} + \frac{\varepsilon}{2}\,.\end{aligned}$$

A similar reasoning for the difference $|x(t,x_0,y_0) - \overline{x}_{2T}(t)|$ on the line segment $[2T,\ 3T]$ gives the estimate

$$\mathbf{E}|x(t,x_0,y_0) - \overline{x}_{2T}|^2 < A^2|y_0|^2 \exp\{-2(2\rho - (RL_2)^2)T\} \le \left(\frac{\varepsilon}{4}\right)^2 \frac{1}{\beta}\left(\frac{1}{4}\right)^2.$$

Here $\overline{x}_{2T}(t)$ is a solution of (2.61) such that $\overline{x}_{2T}(2T) = x(2T, x_0, y_0)$.

Hence,

$$\|x(t,x_0,y_0)\|_V < \frac{\varepsilon}{4}\left(\frac{1}{2}\right)^2 + \frac{\varepsilon}{4}\frac{1}{2} + \frac{\varepsilon}{2}\,.$$

For the line segment $[kT,\ (k+1)T]$, we similarly get

$$\|x(t,x_0,y_0)\|_V < \frac{\varepsilon}{2} + \frac{\varepsilon}{4}\left(\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^k\right).$$

And, hence, for an arbitrary $t \geq 0$, the latter inequality gives

$$\|x(t, x_0, y_0)\|_V < \frac{\varepsilon}{2} + \frac{\varepsilon}{4}\left(\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon\,.$$

Since the norms $\|.\|_2$ and $\|.\|_V$ are equivalent, a use of estimate (2.72) finishes the proof of the theorem. □

The main condition in the theorem we have just proved is existence of a quadratic form having certain properties. If $x \in \mathbf{R}$ and the zero solution of system (2.61) is totally stable, then this condition is satisfied for the function $V(x) = x^2$. Indeed, let there exist $x_0 \in \mathbf{R}$ such that $\dot{V}(x_0) > 0$. Consider a solution of equation (2.61) such that $x(0, x_0) = x_0$. Then $\dot{V}(x(0, x_0)) > 0$ and, hence, $V(x(t, x_0))$ is increasing in t in some neighborhood of $t = 0$. Since the zero solution of system (2.61) is totally stable, there is $T > 0$ such that $\dot{V}(x(T, x_0)) = 0$. This identity means that $2x(T, x_0)X(x(T, x_0), 0) = 0$. Hence, system (2.61) has an equilibrium distinct from zero, which is impossible due to total stability. This proves the following corollary.

Corollary 2.1. *Let $x \in \mathbf{R}$ in system* (2.68)*. Then, if the zero solution of system* (2.61) *is totally stable with non-random initial conditions, then the zero solution of system* (2.68) *is mean square stable with non-random initial conditions uniformly in $t_0 \geq 0$.*

2.6 A study of stability of the invariant set via the reduction principle. Regular perturbations

The stability problem for the invariant set is well studied in the case where the set is stable for the initial conditions lying in some manifold that contains this set. If the invariant set is a point, this problem was solved in [123], and this result is known in the stability theory as the reduction principle. A similar result for the general case is obtained in [139, Ch. 2, § 3].

In this section, we generalize the later result to equations with random perturbations,

$$\frac{dx}{dt} = F(x) + \sigma(t, x)\xi(t)\,, \tag{2.84}$$

where $t \geq 0$, $x \in \mathbf{R}^n$, $\xi(t)$ is a random process, absolutely integrable with probability 1 on every bounded segment of the semiaxis $t \geq 0$. With respect to F and σ, we assume that they are Borel measurable with respect to $t \geq 0$, and are Lipschitz continuous in x with a constant$L$ for $t \geq 0$, $x \in \mathbf{R}^n$. Assume that the system under consideration has an invariant set M in $\mathbf{R}^n$ and it is a subset of a "larger" invariant set N. Let system (2.84) become deterministic on the set N. The problem is to reduce the study of stability of the invariant set M for the system to a study of stability of this set in N, where the system is already deterministic.

Definition 2.9. We say that M is *stable in* N for $t \geq t_0$ if for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that, if x_0 satisfies $\rho(M, x_0) < \delta$, $x_0 \in N$, then

$$\rho(x(t, t_0, x_0), M) < \varepsilon \quad \text{for} \quad t \geq t_0 \,. \tag{2.85}$$

Here $x(t, t_0, x_0)$ is a solution of equation (2.84) such that $x(t, x_0, x_0) = x_0$.

It is clear that stability of M in N does not always imply that this set is stable, if, however, M is unstable on N then the set M is unstable. So the problem is to find conditions that would yield stability of a set M if it is stable in N.

Theorem 2.11. *Let a positively invariant set $N \subset D \subset \mathbf{R}^n$ for system* (2.84)*, where D is a bounded domain, contain a closed, positively invariant set M asymptotically stable on N.*

Let N be a set of the form $V(x) = 0$, $x \in D$, and M be its subset, where $V(x)$ is a positive definite function on $\overline{D}$, Lipschitz continuous on D with a constant B, and let

$$\frac{d^0 V}{dt} \leq -C_1 V \,, \tag{2.86}$$

where $\frac{d^0 V}{dt}$ is the Lyapunov operator along the truncated deterministic system $\frac{dx}{dt} = F(x)$,

$$\|\sigma(t, x)\| \leq C_2 V(x) \,, \tag{2.87}$$

where C_1, C_2 are positive constants and $\| \cdot \|$ is the norm of the matrix.

If there exists a sequence of numbers $\{T_n\}$, $T_n \to \infty$ as $n \to \infty$, such that

$$\mathbf{P}\left\{ \sup_{t \geq 0} \frac{1}{T_n} \int\limits_t^{t+T_n} |\xi(s)| \, ds > \frac{C_1}{BC_2} \right\} \to 0, \ n \to \infty \,, \tag{2.88}$$

then the set M is uniformly in $t_0 \geq 0$ stochastically stable such that for arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta = \delta(\varepsilon_1, \varepsilon_2)$ such that the inequality $\rho(x_0, M) < \delta$ yields the estimate

$$\mathbf{P}\left\{ \sup_{t \geq t_0} \rho(x(t, t_0, x_0), M) > \varepsilon_1 \right\} < \varepsilon_2 \,. \tag{2.89}$$

Proof. Without loss of generality, we will assume that $t_0 = 0$. It follows from condition (2.87) that the restriction of system (2.84) to N is the deterministic system

$$\frac{dx}{dt} = F(x)\,.$$

Denote by $V_{\delta,\mu}$ the set of points in $\mathbf{R}^n$ that belong to the δ-neighborhood of the set M and satisfy the inequality $V(x) \le \mu$, and let $V_{\delta,0} := U_\delta \cap N$, where U_δ is a δ-neighborhood of the set M.

Take arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $U_{\varepsilon_1} \subset D$. Choose $\delta = \delta(\varepsilon_1, \varepsilon_2)$ such that, if $x_0 \in V_{\delta,0}$, then

$$\sup_{t\ge 0} \rho(x(t,x_0), M) < \frac{\varepsilon_1}{2}\,. \tag{2.90}$$

Possibility of such a choice is implied, since M is uniformly stable on N. Now, choose $T > 0$ such that

$$\rho(x(t,x_0), M) < \frac{\delta}{2} \quad \text{for} \quad t \ge T \tag{2.91}$$

if $x_0 \in V_{\delta,0}$. Asymptotic stability implies that this can always be done. Take $T_{n_0} \ge T$ such that

$$\mathbf{P}\left\{ \sup_{t\ge 0} \frac{1}{T_{n_0}} \int\limits_t^{t+T_{n_0}} |\xi(s)|\, ds > \frac{C_1}{BC_2} \right\} < \varepsilon_2\,, \tag{2.92}$$

and denote T_{n_0} by T for convenience.

Let us construct the following sets:

$$I = \left\{ \omega : \sup_{k\ge 0} \frac{1}{T} \int\limits_{kT}^{(k+1)T} |\xi(s)|\, ds > \frac{C_1}{BC_2} \right\} \tag{2.93}$$

and

$$I_k = \left\{ \omega : \frac{1}{T} \int\limits_{kT}^{(k+1)T} |\xi(s)|\, ds > \frac{C_1}{BC_2} \right\}, \tag{2.94}$$

where $k = 0,\ 1, \dots$.

Then, for arbitrary $k \ge 0$, we have the inclusions $I_k \subset I$ and

$$\mathbf{P}\{I\} < \varepsilon_2\,. \tag{2.95}$$

Let $x(t, x_0)$ and $x(t, x_1)$ be solutions of the initial system satisfying the initial conditions $x(0, x_0) = x_0$ and $x(0, x_1) = x_1$. Then the Lipschitz conditions and the Gronwall-Bellman lemma yield the following estimate on the line segment $[0,\ T]$:

$$|x(t, x_0) - x(t, x_1)| \le |x_0 - x_1| e^{Lt} e^{L \int\limits_0^t |\xi(s)| ds} . \tag{2.96}$$

Since

$$|x_0 - x_1| \le d = \frac{\delta}{2} e^{-LT - LT \frac{C_1}{BC_2}} , \tag{2.97}$$

we have that

$$\mathbf{P} \left\{ \sup_{t \in [0, T]} |x(t, x_0) - x(t, x_1)| > \frac{\delta}{2} \right\} < \varepsilon_2 . \tag{2.98}$$

By a lemma in [139, p. 69], the set $V_{\delta,\mu}$ is contracted to $V_{\delta,0}$ for $\mu \to 0$. Hence, there exists $\mu_0 = \mu_0(\delta) > 0$ such that for an arbitrary point $x_1 \in V_{\delta,\mu_0}$ there is a point x_0 in $V_{\delta,0}$ satisfying inequality (2.97). Choose $\mu \le \mu_0$ such that inequality (2.97) would hold on some neighborhood of the point $x_0 \in V_{\delta,0}$ for an arbitrary point x_1 in $V_{\delta,\mu}$. The so constructed open covering of the set $V_{\delta,0}$ contains a finite subcovering, since the set is compact. Let $z_1, ...z_l$ be elements of each set that make a finite subcovering, and which belong to $V_{\delta,0}$. Let y be an arbitrary point in $V_{\delta,\mu}$ and z a point in $V_{\delta,0}$ satisfying (2.97). Then, on the line segment $[0,\ T]$, we have the inequality

$$\mathbf{P} \left\{ \sup_{t \in [0,\ T]} \rho(x(t, y), M) > \varepsilon_1 \right\} \le \mathbf{P} \left\{ \sup_{t \in [0,\ T]} \rho(x(t, y), x(t, z)) > \frac{\varepsilon_1}{2} \right\}$$
$$+ \mathbf{P} \left\{ \sup_{t \in [0,\ T]} \rho(x(t, z), M) > \frac{\varepsilon_1}{2} \right\}$$

and, using (2.90), (2.97), and (2.98), we get the estimate

$$\mathbf{P} \left\{ \sup_{t \in [0,\ T]} \rho(x(t, y), M) > \varepsilon_1 \right\} < \varepsilon_2 . \tag{2.99}$$

Also, by (2.91), we have that

$$\mathbf{P} \left\{ \rho(x(T, y), M) > \delta \right\} < \varepsilon_2 . \tag{2.100}$$

Hence, by (2.99), trajectories of the solution $x(t, y)$ do not leave the domain D with probability greater than $1 - \varepsilon_2$, and using inequality (2.86) and a lemma in [70, pp. 23, 28] we see that $V(x(t, y))$ satisfies the estimate

$$V(x(t, y)) \le V(y) e^{BC_2 \left(\frac{1}{t} \int\limits_0^t |\xi(s)|\, ds - \frac{C_1}{BC_2} \right) t} \tag{2.101}$$

that clearly holds for $\omega \in \overline{I}$, which, by (2.95), implies that the inequality $V(x(T,y)) \le \mu$ holds with probability greater than $1-\varepsilon_2$.

We thus see that, if $\omega \in \overline{I}$, the solution $x(t,y)$ does not leave an ε_1-neighborhood of the set M for $t \in [0,\ T]$ and belongs to $V_{\delta,\mu}$ for $t = T$. It is clear that $\overline{I_0} \subset \{\omega : x(T, y(\omega), \omega) \in V_{\mu,\delta}\}$ and, hence, $\overline{I} \subset \{\omega : x(T, y(\omega), \omega) \in V_{\mu,\delta}\}$.

Now, consider the behavior of the solution $x(t,y)$ on the line segment $[T, 2T]$. Construct a finite-valued random variable $z(\omega)$ that takes values in $V_{\delta,0}$ with probability 1 as follows. Let $B = \{\omega : x(T,y,\omega) \in V_{\mu,\delta}\}$. Evidently, B is measurable. Set $z(\omega) = z_1$ for $\omega \in B$ such that

$$|z_1 - x(T,y,\omega)| \le \min\{|z_2 - x(T,y,\omega)|, \dots, |z_l - x(T,y,\omega)|\}\,, \tag{2.102}$$

and $z(\omega) = z_2$ for $\omega \in B$ such that

$$|z_2 - x(T,y,\omega)| \le \min\{|z_1 - x(T,y,\omega)|,\ |z_3 - x(T,y,\omega)| \dots, |z_l - x(T,y,\omega)|\}\,. \tag{2.103}$$

For other $z_3, \dots z_l$, the values of $z(\omega)$ are defined similarly. The set of such ω is measurable. If $\omega \notin B$, then $z(\omega) = z_{l+1}$, where z_{l+1} is an arbitrary point in $V_{\delta,0}$. The so constructed $z(\omega)$ will be a random variable, since the sets $\{\omega : z(\omega) = z_i\}$, $i = \overline{1, l+1}$, are measurable. It follows from the construction that $\omega \in B$ satisfies the inequality

$$|x(T,y,\omega) - z(\omega)| \le d, \tag{2.104}$$

and, in particular, it holds on the set $\overline{I}$. Note that, since system (2.84) degenerates into a deterministic system on the invariant set N, inequalities (2.90) and (2.91) hold for $x_0 = x_0(\omega)$, which is a random variable, with the same probability as the probability of $x_0(\omega)$ lying in $V_{\delta,0}$. This follows from the definition of stability and a lemma in [139, p. 68] that implies that the limit in the definition of asymptotic stability is uniform in $x_0 \in U_\delta(M) \cap N$.

A similar reasoning gives

$$\mathbf{P}\left\{\sup_{[T,\ 2T]} \rho(x(t,y), M) > \varepsilon_1\right\} \le \mathbf{P}\left\{\sup_{[T,\ 2T]} \rho(x(t,y), x(t,T,z(\omega))) > \frac{\varepsilon_1}{2}\right\} + \mathbf{P}\left\{\sup_{[T,\ 2T]} \rho(x(t,T,z(\omega)), M) > \varepsilon_1\right\},$$

where $x(t,T,z(\omega))$ is a solution of system (2.84) such that $x(T,T,z(\omega)) = z(\omega)$.

And since ω, which belong to the set indicated in the first term of the latter inequality, also belongs to $I_1 \subset I$ and the event in the second term takes place with probability zero, the above inequality leads to the estimates

$$\mathbf{P}\left\{ \sup_{[T,\ 2T]} \rho(x(t,y), M) > \varepsilon_1 \right\} < \varepsilon_2\,, \tag{2.105}$$

$$\mathbf{P}\big\{\rho(x(2T,y), M) > \delta\big\} < \varepsilon_2\,. \tag{2.106}$$

For ω such that $x(t,y) \in D$ if $t \in [T,\ 2T]$, estimate (2.101) holds, which implies that $V(x(2T,y)) \leq \mu$ for $\omega \in \overline{I}$.

Hence, for $\omega \in \overline{I}$, the solution $x(t,y)$ does not leave the ε_1-neighborhood of the set M, and when $t = 2T$, it enters the set $V_{\delta,\mu}$.

A similar argument applied to the following intervals yields that the following inclusion holds for an arbitrary $y \in U_{\delta,\mu}$:

$$\left\{\omega :\ \sup_{t\geq 0} \rho(x(t,y), M) > \varepsilon_1 \right\} \subset I,$$

which, together with (2.95), gives the estimate

$$\mathbf{P}\left\{ \sup_{t\geq 0} \rho(x(t,y), M) > \varepsilon_1 \right\} < \varepsilon_2. \tag{2.107}$$

However, by a lemma in [139, p. 69], the set $U_{\delta,\mu}$ contains some δ_1-neighborhood of the set M. Then, for arbitrary $\varepsilon_1,\ \varepsilon_2$ there exists $\delta_1 = \delta_1(\varepsilon_1,\ \varepsilon_2)$ such that, if $\rho(x(t,y), M) < \delta_1$, then

$$\mathbf{P}\left\{ \sup_{t\geq t_0} \rho(x(t,t_0,y), M) > \varepsilon_1 \right\} < \varepsilon_2,$$

which is sufficient for the set M to be uniformly stochastically stable, by the definition. $\square$

Remark 3. It can be seen from the proof of the theorem that condition (2.88) can be somewhat weakened by replacing it with the supremum over all kT_n, where $k \in Z_+$ and T_n are the terms that enter this condition. However, it is impossible to remove this condition altogether, which shows the following example of the system

$$\begin{aligned} \frac{dx_1}{dt} &= -ax_1\ + \xi(\omega)x_1, \\ \frac{dx_2}{dt} &= -x_2\ + x_1, \end{aligned} \tag{2.108}$$

where $a > 0$, $\xi(\omega)$ is a random variable that takes arbitrarily large positive values. It is clear that the set $x_1 = 0$ is invariant for system (2.108), and the equilibrium $(0,\ 0)$ is asymptotically stable on it. The function $V = |x_1|$ satisfies the conditions of the theorem but regardless of how small the value of $\mathbf{E}|\xi|$ is, the norm of a solution of system (2.108) approaches to infinity with probability not less than $\mathbf{P}\{\xi > a\}$.

Remark 4. It is rather easy to show that if the process $|\xi(t)|$ satisfies the law of large numbers uniformly in t_0, and $\sup_{t\ge 0}\mathbf{E}|\xi(t)| < \frac{C_1}{BC_2}$, then condition (2.88) always holds true.

Remark 5. If the locally integrable process $\xi(t)$, for $t \ge 0$, satisfies the condition

$$\mathbf{P}\left\{\sup_{t\ge T}|\xi(t)| > A\right\} \to 0,\ T \to \infty\,, \tag{2.109}$$

for some positive $A < \frac{C_1}{BC_2}$, then condition (2.88) is also satisfied.

Indeed, let $\frac{C_1}{BC_2} = A + \delta,\ \delta > 0$. Let us show that for an arbitrary $\varepsilon > 0$ there exists T_0 such that the following holds for arbitrary $T \ge T_0$:

$$\mathbf{P}\left\{\sup_{t\ge 0}\frac{1}{T}\int_t^{t+T}|\xi(s)|\,ds \le \frac{C_1}{BC_2}\right\} > 1-\varepsilon\,. \tag{2.110}$$

It is clear that, for any $u > 0$,

$$\begin{aligned}\left\{\omega:\ \sup_{t\ge 0}\frac{1}{T}\int_t^{t+T}|\xi(s)|\,ds \le \frac{C_1}{BC_2}\right\} &= \left\{\omega:\ \sup_{t\le u}\frac{1}{T}\int_t^{t+T}|\xi(s)|\,ds \le \frac{C_1}{BC_2}\right\}\\ &\quad\bigcap\left\{\omega:\ \sup_{t\ge u}\frac{1}{T}\int_t^{t+T}|\xi(s)|\,ds \le \frac{C_1}{BC_2}\right\}.\end{aligned} \tag{2.111}$$

By (2.109), for the chosen ε there exists T such that

$$\mathbf{P}\left\{\sup_{t\ge T}|\xi(t)| \le A\right\} > 1-\frac{2\varepsilon}{5}\,. \tag{2.112}$$

Choose $T_0 \ge T$ such that

$$\mathbf{P}\left\{\frac{1}{T_0}\int_0^{T}|\xi(s)|\,ds \le \delta\right\} > 1-\frac{\varepsilon}{5}\,. \tag{2.113}$$

This can be done, since $\xi(t)$ is locally integrable.

Let $t \le T$. Then, for arbitrary $T_1 \ge T_0$,

$$
\begin{aligned}
&\left\{\omega : \sup_{t\le T} \frac{1}{T_1} \int_t^{t+T_1} |\xi(s)|\, ds \le \frac{C_1}{BC_2}\right\} \\
&\quad = \left\{\omega : \sup_{t\le T} \frac{1}{T_1} \Big(\int_t^{T} |\xi(s)|\, ds \ + \int_T^{t+T_1} |\xi(s)|\, ds\Big) \le \frac{C_1}{BC_2}\right\} \\
&\quad \supset \left\{\omega : \frac{1}{T_1} \Big(\int_0^{T} |\xi(s)|\, ds \ + \sup_{t\le T} \frac{1}{T_1} \int_T^{t+T_1} |\xi(s)|\, ds\Big) \le \frac{C_1}{BC_2}\right\} \\
&\quad \supset \left(\left\{\omega : \frac{1}{T_1} \int_0^{T} |\xi(s)|\, ds \le \delta\right\} \bigcap \left\{\omega : \sup_{t\le T} \frac{1}{T_1} \int_T^{t+T_1} |\xi(s)|\, ds \le A\right\}\right).
\end{aligned}
\tag{2.114}
$$

However, the probability of the first event in the latter intersection, by (2.113), is greater than $1 - \frac{\varepsilon}{5}$, and the probability of the second event satisfies the estimate

$$
\begin{aligned}
&\mathbf{P}\left\{\omega : \sup_{t\le T} \frac{1}{T_1} \int_T^{t+T_1} |\xi(s)|\, ds \le A\right\} \\
&\qquad \ge \mathbf{P}\left\{\omega : \sup_{t\ge T} |\xi(s)| \le A\right\} > \ 1 - \frac{2\varepsilon}{5}\,.
\end{aligned}
\tag{2.115}
$$

Hence, the probability of the intersection of the events in (2.114) is greater than $1 - \frac{3\varepsilon}{5}$. If $t \ge T$, then

$$
\begin{aligned}
&\mathbf{P}\left\{\omega : \sup_{t\ge T} \frac{1}{T_1} \int_t^{t+T_1} |\xi(s)|\, ds \le \frac{C_1}{BC_2}\right\} \\
&\qquad \ge \mathbf{P}\left\{\omega : \sup_{t\ge T} |\xi(t)| \le A\right\} > \ 1 - \frac{2\varepsilon}{5}\,.
\end{aligned}
$$

Formula (2.111), for $u = T$ and arbitrary $T_1 \ge T_0$, gives

$$
\mathbf{P}\left\{\sup_{t\ge 0} \frac{1}{T_1} \int_t^{t+T_1} |\xi(s)|\, ds \le \frac{C_1}{BC_2}\right\} > 1 - \varepsilon\,,
$$

which implies condition (2.88). Note that relations (2.109) always hold for classes of stochastic processes generated by stochastic differential equations that have zero solution asymptotically stochastically stable, see. e.g. [186, p. 124] that also contains a fairly large bibliography on this matter.

Remark 6. It can be seen from the theorem that, in fact, a stronger stability takes place, i.e., inequality (2.89) holds even if $x_0 = x_0(\omega)$ is a random variable that takes values in a δ-neighborhood of the set M with probability 1. So the stability the theorem deals with takes place for random initial conditions too.

Note that the conditions imposed on the Lyapunov function, in particular the Lipschitz condition, are local, as opposed to theorems in [70, pp. 45–46] where these conditions are global. This significantly simplifies construction of a Lyapunov function with the needed properties.

Example. Let us study the stochastic stability of zero solution of the system

$$\begin{aligned} \frac{dx_1}{dt} &= -x_1 + \sigma_1(t, x_1, x_2)\xi_1(t), \\ \frac{dx_2}{dt} &= x_1 - x_2 + \sigma_2(t, x_1, x_2)\xi_2(t)\,, \end{aligned} \tag{2.116}$$

where

$$|\sigma_1| \le C_1|x_1|, \ |\sigma_2| \le C_2|x_1|, \tag{2.117}$$

and the random process $(\xi_1(t), \xi_2(t))$ satisfies condition (2.88). Note that the stability theorem based on the first order approximation can not be applied here due to (2.117). However, system (2.116) has an invariant straight line, $x_1 = 0$, that contains the invariant set $M = \{(0,0)\}$. On this line, system (2.116) has the form

$$\frac{dx_2}{dt} = -x_2.$$

This gives asymptotic stability of M in the straight line $x_1 = 0$. It is clear that the Lyapunov function $V = |x_1|$ satisfies all conditions of the theorem. By applying the theorem, we see that the zero solution of system (2.116) is uniformly stochastically stable.

2.7 Stability of invariant sets and the reduction principle for Ito type systems

In the previous section, we gave an abstract reduction principle for using in the stability theory of differential systems with regular perturbations in the right-hand side. In this section, we will obtain a similar result for Ito type systems of the form

$$dx = b(x)dt + \sum_{r=1}^{k} \sigma_r(t,x)dW_r(t)\,, \tag{2.118}$$

where $t \geq 0,\ x \in \mathbf{R}^n,\ W_r(t)(r = \overline{1,k})$ are independent Wiener processes.

We assume that the n-dimensional vectors $b(x), \sigma_1(s,x), \ldots, \sigma_k(s,x)$ are continuous in (s,x).

We will study stability of a positively invariant set S_t that belongs to a larger invariant set N_t such that system (2.118) on it becomes deterministic, which will reduce the study of stability of the stochastic system to that of a deterministic system.

Theorem 2.12. *Let a positively invariant set $N \subset D \subset \mathbf{R}^n$ for system* (2.118)*, where D is a bounded domain, contain a closed, positively invariant subset S asymptotically stable in N. Let also there exist $L > 0$ such that, for arbitrary $x, y \in \mathbf{R}^n$,*

$$|b(x) - b(y)| \leq L|x - y|\,, \tag{2.119}$$

and the following conditions hold in every cylinder $\{t \geq 0\} \times \{|x| < r\}$:

$$\sum_{r=1}^{k} |\sigma_r(t,x) - \sigma_r(t,y)| \leq B_r|x - y|\,,$$

$$\sum_{r=1}^{k} |\sigma_r(t,x)| \leq B_r(1 + |x|)\,.$$

If N is a set of the form $V(x) = 0,\ x \in D$, where $V(x)$ is a nonnegative definite, twice continuously differentiable function on $\mathbf{R}^n$, satisfying the conditions

$$LV = (\nabla V, b(x)) + \frac{1}{2}\sum_{r=1}^{k}(\nabla, \sigma_r(t,x))^2 V \leq -C_1 V, \tag{2.120}$$

$$V_r = \inf_{|x|>r} V(x) \to \infty,\ r \to \infty, \qquad ||\sigma(t,x)||^2 \leq C_2 V(x)\,, \tag{2.121}$$

on $\mathbf{R}^n$, where C_1, C_2 are positive constants, σ is a matrix with columns σ_r, $\nabla = (\frac{\partial}{\partial x_1}, \cdots \frac{\partial}{\partial x_n})$, then the set S is uniformly stochastically stable, and for arbitrary $\varepsilon_1 > 0,\ \varepsilon_2 > 0$ there exists $\delta = \delta(\varepsilon_1, \varepsilon_2)$ such that, for $\rho(x_0, S) < \delta$, we have the inequality

$$\mathbf{P}\left\{\sup_{t \geq t_0} \rho(x(t,t_0,x_0), S) > \varepsilon_1\right\} < \varepsilon_2\,. \tag{2.122}$$

Remark 7. The second condition in (2.119) can be dropped if the function σ_r is assumed to be linearly bounded and Lipschitz continuous with a unique constant on B rather than for every cylinder.

Proof. Let us remark that conditions of the theorem and a result in [70, p. 141] imply that equation (2.118) has a unique strong solution for $t \geq t_0$ with the initial conditions $x(t_0) = x_0(\omega)$, where $x_0(\omega)$ is an arbitrary random variable independent of the process $W_r(t) - W_r(t_0)$ that satisfies the inequality

$$\mathbf{E}V(x(t, t_0, x_0)) \leq \mathbf{E}V(x_0) \exp\{C_1(t - t_0)\}. \tag{2.123}$$

Without loss of generality, we will assume that $t_0 = 0$. Condition (2.121) implies that system (2.118) becomes the following deterministic system on the set N:

$$\frac{dx}{dt} = b(x). \tag{2.124}$$

Denote by $V_{\delta,\mu}$ a set of points of $\mathbf{R}^n$ belonging to the δ-neighborhood of the set S and satisfying the inequality $V(x) \leq \mu$, and let $V_{\delta,0} := U_\delta \cap N$, where U_δ is a closed δ-neighborhood of the set S.

Take arbitrary $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that $U_{\varepsilon_1}(S) \subset D$. Choose $\delta = \delta(\varepsilon_1, \varepsilon_2)$ such that

$$\sup_{t \geq 0} \rho(x(t, x_0), S) < \frac{\varepsilon_1}{2}. \tag{2.125}$$

Here $x(0, x_0) = x_0 \in V_{\delta,0}$, x_0 is nonrandom. This can be done, since S is uniformly stable on N.

Now, choose $T > 0$ such that

$$\rho(x(t, x_0), S) < \frac{\delta}{2} \tag{2.126}$$

for $t \geq T$ and $x_0 \in V_{\delta,0}$. This can also be achieved, because of the asymptotic stability.

Let $x(t, y)$ be a solution of system (2.118) such that $x(0, y) = y \in V_{\delta,\mu}$, where y is a nonrandom vector.

Consider the function

$$W(t, x) = V(x) \exp\{C_1 t\}, \tag{2.127}$$

and let the generating operator $L_t = \frac{\partial}{\partial t} + L$ act on it. By (2.120), we get

$$L_t W(t, x) = \exp\{C_1 t\} L V(x) + C_1 W(t, x) \leq 0. \tag{2.128}$$

It also follows from (2.123) that

$$\mathbf{E}W(t,x(t,y)) \le V(y)\exp\{2C_1 t\}\,. \tag{2.129}$$

Applying Itos's formula to the process $W(t,x(t,y))$ for $s \le t$ we get

$$W(t,x(t,y)) - W(s,x(s,y)) = \int_s^t L_t W(u,x(u,y))du + \sum_{r=1}^k \int_s^t (\sigma_r(u,x(u,y))dW_r(u).$$

Taking conditional expectation of both sides with respect to the σ-algebra F_s, which enters the definition of a solution of system (2.118), and using (2.128) we get

$$\mathbf{E}(W(t,x(t,y))|F_s) \le W(s,x(s,y))\,.$$

We have used properties of stochastic Ito integral and the fact that $x(s,y)$ is measurable with respect to the σ-algebra F_s. The latter identity implies that the process $W(t,x(t,y))$ is a supermartingale with respect to the family F_t of σ-algebras. Hence, $EW(x(t,y)) \le V(y)$. This proves the inequality

$$\mathbf{E}V(x(t,y)) \le V(y)\exp\{-C_1 t\}\,. \tag{2.130}$$

For an arbitrary natural ν, consider the sets

$$A_\nu = \left\{\omega \in \Omega : \sup_{1\le l\le \nu} V(x(lT,y,\omega)) > \mu\right\}. \tag{2.131}$$

Properties of supermartingales [53, p. 137] imply that

$$\begin{aligned}\mathbf{P}\{A_\nu\} &\le \frac{2\sup_{1\le l\le\nu}\mathbf{E}V(x(lT,y))}{\mu} \\ &\le \frac{2V(y)\exp\{-TC_1\}}{\mu} \le 2\exp\{-TC_1\}\,.\end{aligned} \tag{2.132}$$

By passing to the limit in (2.132) as $\nu \to \infty$ and using that the sequence of sets A_ν is nondecreasing, we obtain

$$\mathbf{P}\left\{\omega : \sup_{l\ge 1} V(x(lT,y)) > \mu\right\} \le 2\exp\{-TC_1\}. \tag{2.133}$$

Denote by A_0 the set

$$A_0 = \left\{\omega : \sup_{l\ge 1} V(x(lT,y)) > \mu\right\}.$$

Let now $x(t,x_0)$ and $x(t,x_1)$ be two solutions of system (2.118) with the initial conditions $x(0,x_0)=x_0\in V_{\delta,0}$ and $x(0,x_1)=x_1\in V_{\delta,\mu}$. Then, by (2.118),

$$x(t,x_1) \;=\; x_1 + \int_0^t b(x(s,x_1))ds + \sum_{r=1}^k \int_0^t \sigma_r(s,x(s,x_1))dW_r(s)\,, \quad (2.134)$$

and, by (2.124),

$$x(t,x_0)=x_0+\int_0^t b(x(s,x_0))ds\,. \quad (2.135)$$

By subtracting (2.135) from (2.134), using the Lipschitz continuity and the Gronwall-Bellman lemma we get the following inequality that holds on the line segment $[0,T]$:

$$\begin{aligned}|x(t,x_1)-x(t,x_0)| \le |x_1-x_0| + L\int_0^t |x(s,x_1)-x(s,x_0)|\,ds \\ + \sum_{r=1}^k \Big|\int_0^t \sigma_r(s,x(s,x_1))dW_r(s)\Big|\,,\end{aligned}$$

which implies that

$$\begin{aligned}\sup_{t\in[0,T]} |x(t,x_1)-x(t,x_0)| \le \Big(|x_1-x_0| \\ +\sum_{r=1}^k \sup_{t\in[0,T]}\Big|\int_0^t \sigma_r(s,x(s,x_1))dW_r(s)\Big|\Big)\exp\{LT\}\,. \quad (2.136)\end{aligned}$$

Applying the same reasoning to the rest of line segments of the form $[\nu T,(\nu+1)T]$, where ν is natural, we get

$$\begin{aligned}\sup_{t\in[\nu T,(\nu+1)T]} |x(t,x_1)-x(t,x_0)| \le \Big(|x(\nu T,x_1)-x(\nu T,x_0)| \\ +\sum_{r=1}^k \sup_{t\in[\nu T,\,(\nu+1)T]}\Big|\int_{\nu T}^t \sigma_r(s,x(s,x_1))dW_r(s)\Big|\Big)\exp\{LT\}\,. \quad (2.137)\end{aligned}$$

Then we have

$$\mathbf{P}\left\{\sup_{\nu\in Z_+}\sum_{r=1}^{k}\sup_{t\in[\nu T,(\nu+1)T]}\left|\int\limits_{\nu T}^{t}\sigma_r(s,x(s,x_1))dW_r(s)\right|>\frac{\delta}{4}\exp\{-LT\}\right\}$$

$$\leq\sum_{\nu=0}^{\infty}\mathbf{P}\left\{\sum_{r=1}^{k}\sup_{t\in[\nu T,(\nu+1)T]}\left|\int\limits_{\nu T}^{t}\sigma_r(s,x(s,x_1))dW_r(s)\right|>\frac{\delta}{4}\exp\{-LT\}\right\}$$

$$\leq\sum_{\nu=0}^{\infty}\mathbf{P}\left\{\sum_{r=1}^{k}\left(\sup_{t\in[\nu T,\,(\nu+1)T]}\left|\int\limits_{\nu T}^{t}\sigma_{1r}(s,x(s,x_1))dW_r(s)\right|\right.\right.$$

$$\left.\left.+\cdots+\sup_{t\in[\nu T,\,(\nu+1)T]}\left|\int\limits_{\nu T}^{t}\sigma_{nr}(s,x(s,x_1))dW_r(s)\right|\right)>\frac{\delta}{4}\exp\{-LT\}\right\}$$

$$\leq\sum_{\nu=0}^{\infty}\sum_{j,i=1}^{n,k}\mathbf{P}\left\{\sup_{t\in[\nu T,(\nu+1)T]}\left|\int\limits_{\nu T}^{t}\sigma_{ij}(s,x(s,x_1))dW_i(s)\right|>\frac{\delta}{4kn}\exp\{-LT\}\right\}$$

$$\leq\sum_{\nu=0}^{\infty}\sum_{j,i=1}^{n,k}k^2n^2 16\frac{\int\limits_{\nu T}^{(\nu+1)T}\mathbf{E}(\sigma_{ij}(t,x(t,x_1)))^2\,dt}{\delta^2}\exp\{2LT\}$$

$$\leq\sum_{\nu=0}^{\infty}\frac{k^2n^2 16\int\limits_{\nu T}^{(\nu+1)T}\mathbf{E}||\sigma(t,x(t,x_1))||^2\,dt}{\delta^2}\exp\{2LT\}$$

$$=\frac{k^2n^2 16\exp\{2LT\}}{\delta^2}\int\limits_{0}^{\infty}\mathbf{E}||\sigma(t,x(t,x_1))||^2\,dt$$

$$\leq\frac{16C_2k^2n^2\exp\{2LT\}}{\delta^2}\int\limits_{0}^{\infty}\mathbf{E}V(x(t,x_1))\,dt\,.$$

The latter inequality holds by (2.121). Using (2.130) we get

$$\mathbf{P}\left\{\sup_{\nu\in Z_+}\sum_{r=1}^{k}\sup_{t\in[\nu T,(\nu+1)T]}\left|\int\limits_{\nu T}^{t}\sigma_r(s,x(s,x_1))dW_r(s)\right|>\frac{\delta}{4}\exp\{-LT\}\right\}$$

$$\leq \frac{16C_2k^2n^2\exp\{2LT\}}{\delta^2}\int_0^\infty \mathbf{E}V(x(t,x_1))dt \leq \frac{16C_2k^2n^2\exp\{2LT\}}{\delta^2}$$

$$\times \int_0^\infty V(x_1)\exp\{-C_1t\} \leq \frac{16C_2k^2n^2\mu\exp\{2LT\}}{\delta^2}\frac{1}{C_1}. \tag{2.138}$$

Denote

$$B = \sup_{\nu\in Z_+}\sum_{r=1}^{k}\sup_{t\in[\nu T,\,(\nu+1)T]}\left|\int_{\nu T}^{t}\sigma_r(s,x(s,x_1))dW_r(s)\right|.$$

Choose T so large and μ so small that, together with (2.126), the following inequalities would be satisfied:

$$\exp\{2LT\}\frac{16C_2k^2n^2\mu}{\delta^2C_1} < \frac{\varepsilon_2}{2}, \tag{2.139}$$

$$2\exp\{-T\} < \frac{\varepsilon_2}{2}. \tag{2.140}$$

If

$$|x_1 - x_0| < d = \frac{\delta}{4}\exp\{-LT\}, \tag{2.141}$$

formula (2.136) gives

$$\mathbf{P}\left\{\sup_{t\in[0,T]}|x(t,x_1) - x(t,x_0)| > \frac{\delta}{2}\right\} < \frac{\varepsilon_2}{2}. \tag{2.142}$$

According to a lemma in [139, p. 69], the set $V_{\delta,\mu}$ can be uniformly contracted to $V_{\delta,0}$ as $\mu \to 0$. Hence, it is possible to find $\mu_0 = \mu_0(\delta) > 0$ such that for any point $x_1 \in V_{\delta,\mu_0}$ there exists a point x_0 in $V_{\delta,0}$, which implies that inequality (2.141) holds.

Choose now $\mu \leq \mu_0$ such that inequality (2.141) would hold for an arbitrary point $x_1 \in V_{\delta,\mu}$ in a neighborhood of the point $x_0 \in V_{\delta,0}$.

Thus constructed covering of the set $V_{\delta,0}$, since it is compact, contains a finite subcovering. Let $z_1, \cdots z_m$ be elements of each set that form the finite subcovering and contained in $V_{\delta,0}$.

Let y be an arbitrary point of $V_{\delta,\mu}$, and z a point of $V_{\delta,0}$ that make the inequality (2.141) hold.

Then, on the line segment $[0,\, T]$, we have

$$\mathbf{P}\left\{\sup_{t\in[0,T]}\rho(x(t,y),S)>\varepsilon_1\right\}\leq \mathbf{P}\left\{\sup_{t\in[0,T]}|(x(t,y)-x(t,z)|>\frac{\varepsilon_1}{2}\right\}$$
$$+\mathbf{P}\left\{\sup_{t\in[0,T]}\rho(x(t,z),S)>\frac{\varepsilon_1}{2}\right\}.$$

This inequality, due to (2.125), (2.141), and (2.142), leads to the estimate

$$\mathbf{P}\left\{\rho(x(t,y),S)>\varepsilon_1\right\}\leq\varepsilon_2\,, \tag{2.143}$$

and, by (2.126), we have

$$\mathbf{P}\left\{\rho(x(T,y),S)>\delta\right\}<\varepsilon_2\,. \tag{2.144}$$

The event opposed to the one in (2.144) clearly takes place for $\omega\in\Omega$ that belong to the complement of the event

$$\left\{\omega:\sup_{l\geq 1}V(x(lT,y))>\mu\right\}\cup\left\{\omega:B>\frac{\delta}{4}\exp\{-LT\}\right\}.$$

Denote this complement by I. Then

$$\mathbf{P}\left\{I\right\}\geq 1-\varepsilon_2\,. \tag{2.145}$$

For $\omega\in I$, we also have the inequality $V(x(T,y))\leq\mu$. Hence, for $\omega\in I$, the solution $x(t,y)$ of system (2.118) does not leave the ε_1-neighborhood of the set S for $t\in[0,\, T]$, and belongs to $V_{\delta,\mu}$ for $t=T$. Thus, using (2.145) we see that this can happen with probability not less than $1-\varepsilon_2$.

Now, consider the behavior of the solution $x(t,y)$ on the line segment $[T,2T]$. Construct a finite-valued random variable $z(\omega)$ that takes values in $V_{\delta,0}$ with probability 1 as follows.

Let $C=\{\omega:x(T,y,\omega)\in V_{\delta,\mu}\}$. It is clear that the set C is F_T-measurable. Set $z(\omega)=z_1$ for $\omega\in C$ such that

$$|z_1-x(T,y,\omega)|\leq min\{|z_2-x(T,y,\omega)|,\cdots|z_m-x(T,y,\omega)|\}\,. \tag{2.146}$$

The set formed by such ω is F_T-measurable. For other $z_2,\cdots z_m$, the values $z(\omega)$ are defined in a similar way. If $\omega\notin C$, then $z(\omega)=z_{m+1}$, where z_{m+1} is an arbitrary point in $V_{\delta,0}$. Since the sets $\{\omega:z(\omega)=z_i\}$, $i=\overline{1,m+1}$, are measurable, such constructed $z(\omega)$ will be a F_T-measurable random variable. It follows from its construction that, for $\omega\in C$, we have the inequality

$$|x(T,y,\omega)-z(\omega)|\leq d\,. \tag{2.147}$$

In particular, it holds for $\omega \in I$.

Note that, since system (2.118) becomes deterministic on the invariant set N, inequalities (2.125) and (2.126) hold for the random variable $x_0 = x_0(\omega)$ with the same probability as the probability of $x_0(\omega)$ belongs to $V_{\delta,0}$. This follows from the definition of stability and a lemma in [139, p. 62] that implies uniformity in $x_0 \in U_\delta(S) \cap N$ of the limit in the definition of asymptotic stability.

A similar reasoning leads to the estimate

$$\mathbf{P}\left\{\sup_{t\in[T,2T]} \rho(x(t,y),S) > \varepsilon_1\right\} \le \mathbf{P}\left\{\sup_{t\in[T,\,2T]} |(x(t,y) - x(t,T,z(\omega))| > \frac{\varepsilon_1}{2}\right\} + \mathbf{P}\left\{\sup_{t\in[T,\,2T]} \rho(x(t,T,z(\omega)),S) > \frac{\varepsilon_1}{2}\right\},$$

where $x(t,T,z(\omega))$ is a solution of system (2.118) such that $x(T,T,z(\omega)) = z(\omega)$.

Since inequality (2.147) holds for $\omega \in I$ and the event in the second term has zero probability, the above estimate yields the estimate

$$\mathbf{P}\left\{\sup_{t\in[T,\,2T]} \rho(x(t,y),S) > \varepsilon_1\right\} < \varepsilon_2\,. \tag{2.148}$$

Moreover, for $t = 2T$, we have

$$\mathbf{P}\left\{\rho(x(2T,y),S) > \delta\right\} < \varepsilon_2\,. \tag{2.149}$$

Since

$$\left\{\omega:\ V(x(2T,y)) > \mu\right\} \subset \left\{\omega: \sup_{l\ge1} V(x(lT,y)) > \mu\right\} \subset \overline{I}\,,$$

the solution $x(t,y)$ does not leave an ε_1-neighborhood of the set S for $\omega \in I$, and belongs to the set $V_{\delta,\mu}$ for $t = 2T$.

A similar argument applied to subsequent intervals prove that the following inclusion holds for an arbitrary $y \in V_{\delta,\mu}$:

$$\left\{\omega:\ \sup_{t\ge0} \rho(x(t,y),S) > \varepsilon_1\right\} \subset \overline{I}$$

which, with a use of (2.145), gives the estimate

$$\mathbf{P}\left\{\omega:\ \sup_{t\ge0} \rho(x(t,y),S) > \varepsilon_1\right\} < \varepsilon_2\,. \tag{2.150}$$

By a lemma in [139, p. 69], the set $V_{\delta,\mu}$ contains some δ_1-neighborhood of the set S and, hence, for arbitrary $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exists $\delta_1 = \delta_1(\varepsilon_1, \varepsilon_2)$ such that

$$\mathbf{P}\left\{\sup_{t \geq t_0} \rho(x(t, t_0, y), S) > \varepsilon_1\right\} < \varepsilon_2$$

for $\rho(y, S) < \delta_1$, which proves asymptotic stability of the set S. □

At the end of this subsection, we give an example that illustrates the above theorem.

Example. Let us study stability of the zero solution of the system

$$\begin{aligned} dx_1 &= -x_1 dt + \sigma_1(t, x_1, x_2) dW_1(t)\,, \\ dx_2 &= (f(x_1) - x_2) dt + \sigma_2(t, x_1, x_2) dW_2(t)\,, \end{aligned} \tag{2.151}$$

where $f(0) = 0$, f is Lipschitz continuous on $\mathbf{R}^1$ with a constant L, and $|\sigma_1| \leq C_1|x_1|$, $|\sigma_2| \leq C_2|x_1|$.

Note that the above conditions again do not permit to study stability using the theorem on first order approximation. However, system (2.151) has an invariant set, $x_1 = 0$, that contains the point $\{0; 0\}$.

Restricted to the invariant set, system (2.151) has the form

$$dx_2 = -x_2 dt\,,$$

which implies asymptotic stability of the zero solution in the set $x_1 = 0$. Take $V = x_1^2$ to be the Lyapunov function. Then the generating operator of system (2.151) satisfies

$$LV = -2x_1^2 + \sigma_1^2(t, x_1, x_2) \leq -2x_1^2 + C_1^2 x_1^2 \leq (-2 + C_1^2) V\,.$$

If $C_1 \leq \sqrt{2}$, all conditions of the theorem are satisfied. This implies that the equilibrium $(0; 0)$ of system (2.151) is uniformly stochastically stable.

Note that if we take the Lyapunov function to be $V = x_1^2 + x_2^2$ and apply a theorem from [70, p. 207], then the stochastic stability will take place only if a stricter condition is imposed, namely,

$$L + \max\{C_1^2 + C_2^2\} \leq 2\,.$$

2.8 Comments and References

Section 2.1. The idea of using the method integral manifolds first appeared in the theory of differential equations in the work of Bogolyubov [19], where an idea was proposed to not consider one particular solution of the differential equation but a set of such solutions. It often happens that a family of integral curves forms a surface that has a simple enough topological structure and to study which is simpler that to integrate the initial system. From the analytical point of view, a system having an invariant set admits on this set a reduction of the order of the system, which significantly simplifies the study. After appearance of this work, the method of integral manifolds undergoes an intensive and brisk development in the works of Bogolyubov and Mitropol'sky [20, 21], Mitropol'sky and Lykova [104], Pliss [123], Samoilenko [139], and others, as well as in the works of Fenichel [44], Moser [115], Sacker [132, 133], Sacker and Sell [134, 135], Sell [153, 154]. Invariant sets were studied with a use of Lyapunov functions by Ignatiev in [59, 60].

Later it turned out that the method of integral sets can be successfully applied not only to systems of ordinary differential equations but also to other classes of systems. This was done for impulsive systems by Samoilenko and Perestyuk [143], for integral-differential equations by Filatov [45], for functional-differential and difference equations by Mitropol'sky, Samoilenko, and Martynyuk [107], Sharkovsky, Pelyukh [121], for infinite dimensional systems by Samoilenko and Teplinsky [152].

For equations with random perturbations, the theory of invariant sets has not been completely developed. This is due, first of all, to the fact that a solution of such a system is a random process for which a deterministic surface must be invariant, and thus it is difficult to obtain such conditions in a general case. Especially, this is the case when the perturbations in the right-hand side are regular. Special results in this direction, in particular, questions related to stability of invariant sets are treated by Khominsky [70, p. 323]. Random invariant sets, in particular, attractors of random dynamic systems were studied by Arnol'd in [8]; one can also find there a fairly large bibliography; see also Baxler [25], Carverhill [32], Mohammed and Sceeutzowgac [110], Wanner [194], Waymire and Duan [196]. The results given here are published by the authors in [147, 164, 163].

Section 2.2. Random invariant sets, considered as attractors of random dynamical systems generated by stochastic Ito equations were considered by Arnol'd in the mentioned monograph [8]. Since solutions of stochastic Ito systems are Markov processes, this permits to study them by using a well developed analytical machinery of the theory of Markov processes, see Kolmogorov [74], Dub [42], Dynkin [43]. This approach was used to study nonrandom invariant sets by Kulinich and Babchuk [14, 15], Kulinich and Denisova [39]

Kulinich and Pereguda [88], where conditions for invariance of level lines of some function $G(t, x)$ were obtained in terms of a generating operator of the Markov process. The authors there also gave a description of classes of stochastic systems for which given sets are invariant, as well as conducted a study of the behavior of the solution on invariant sets. In this connection, we also need to mention the work of Gikhman and Klichkova [50], which contains a construction of a system of stochastic equations such that a given set is invariant. Invariant sets for systems with stochastic Stratonovich integral were studied by Abraham, Marsden and T. Ratiu [1]. Deterministic invariant sets were also considered by Aubin and Da Prato [120], Da Prato, Frankovsra [34, 35], Aubin, Doss [13], Buckdahn, Quincampoix, Rainer, Teichmann [29] see also Filipovic [46], Filipovic, Tappe and Teichmann [47], Tappe [181] Zabczyk [197]. The material of this section is published by the authors in [147, 169].

Section 2.3. The behavior of invariant sets of deterministic dynamic systems with small perturbations were studied by numerous authors, e.g., Samoilenko [139], Chueshov [33], where an extensive bibliography is included. For systems with random perturbations, some results in this directions are contained in the monograph of Arnold [8]. The results included in this section are obtained by the authors in [147, 163].

Sections 2.4–2.5. The idea of the reduction principle first appeared in works of Poincare [126], and a rigorous mathematical justification was given by Pliss [124]. It was shown there that a study of stability of the zero solution can be reduced to a study of its behaviour on an invariant manifold on which the dimension of the system is less than the dimension of the whole space.

For systems with random perturbations, this principle is especially important, since it permits to reduce the study of stability not only to that of a system of a smaller order but also to systems that become deterministic on the invariant manifold. The stability problem for stochastic system can thus be reduced to the same problem for a deterministic system. Some related results are contained in the monograph of Tsar'kov [186, p. 393] and in the work of Korolyuk [80], where the study of stability of a stochastic system is conducted by considering stability of a specially averaged deterministic system. The results in this section are obtained by the authors in [148, 147].

Sections 2.6–2.7. A generalization of the reduction principle, which allows to study manifolds of a more general nature than equilibriums, was obtained by Samoilenko [138]. The authors do not know similar results for systems with random perturbations. The material in this section was published by the authors in [148, 167].

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Chapter 3

Linear and quasilinear stochastic Ito systems

In this chapter we will be dealing with qualitative analysis of the behavior of solutions of linear and weakly nonlinear stochastic Ito systems with variable coefficients.

In Section 3.1, we introduce a notion of exponential mean square dichotomy for linear stochastic Ito systems and find a relation between the dichotomy and the existence of solutions, which are mean square bounded on the semiaxis, of corresponding nonhomogeneous systems.

Section 3.2 deals with a study of exponential dichotomy using sign indefinite quadratic forms. Here we show that a sufficient condition for dichotomy is the existence of a form such that the corresponding differential operator along the system is a negative definite quadratic form.

In Sections 3.3 – 3.4, we study existence conditions for solutions, which are mean square bounded on the axis, of linear and weakly linear stochastic Ito systems. This study is conducted in terms of Green's function of the linear part, and is used to obtain an integral representation for the corresponding solution.

In Section 3.5, we generalize the notion of a solution of a stochastic equation by making it agree with the corresponding flow of homeomorphisms. This permits in the case under consideration to prove that the dichotomy is equivalent to that the nonhomogeneous equation has solutions that probability bounded on the axis.

Sections 3.6 – 3.7 contain an asymptotic study of stochastic systems in terms of a construction of a certain deterministic system that is asymptotically equivalent to the initial stochastic system, which means that the difference between the corresponding solutions tends to zero in square mean, or with probability 1 as $t \to \infty$. This allows to reduce the study of the stochastic object to a more simple deterministic one.

3.1 Mean square exponential dichotomy

Consider the system of linear differential stochastic Ito equations

$$dx = A(t)xdt + \sum_{i=1}^{m} B_i(t)xdW_i(t)\,, \tag{3.1}$$

where $t \geq 0$, $x \in \mathbf{R}^n$, $A(t)$, $B_i(t)$ are deterministic matrices, continuous and bounded on the positive semiaxis, $W_i(t), i = \overline{1,\ldots m}$, are scalar Wiener processes, totally independent, defined on a probability space (Ω, F, P). As is known, see e.g. [186, p. 230], for an arbitrary $x_0 \in \mathbf{R}^n$, system (3.1) has a unique strong solution of the Cauchy problem, $x(t, x_0)$, $x(0, x_0) = x_0$, defined for $t \geq 0$ and having finite second moments for $t \geq 0$.

Definition 3.1. System (3.1) is called *mean square exponentially dichotomous* on the semiaxis $t \geq 0$ if the space $\mathbf{R}^n$ can be represented as a direct sum of two subspaces, R^- and R^+, such that an arbitrary solution $x(t, x_0)$ of system (3.1) such that $x_0 \in R^-$ satisfies

$$\mathbf{E}|x(t, x_0)|^2 \leq K \exp\{-\gamma(t-\tau)\}\mathbf{E}|x(\tau, x_0)|^2 \tag{3.2}$$

for $t \geq \tau \geq 0$, and an arbitrary solution $x(t, x_0)$ of system (3.1) such that $x_0 \in R^+$ satisfies the inequality

$$\mathbf{E}|x(t, x_0)|^2 \geq K_1 \exp\{\gamma_1(t-\tau)\}\mathbf{E}|x(\tau, x_0)|^2 \tag{3.3}$$

for $t \geq \tau \geq 0$ and arbitrary $\tau \geq 0$, where K, K_1, γ, γ_1 are some positive constants independent of τ, x_0.

The mean square exponentially stable system (3.1) is an example of such a system; here $R^+ = \{0\}$ and $R^- = \mathbf{R}^n$.

As it follows from the works mentioned above, the questions of exponential dichotomy on the semiaxis for ordinary differential equations is equivalent to the problem of the nonhomogeneous system having solutions that are bounded on the semiaxis. Similar questions for system (3.1) will be studied in this section.

In what follows, without loss of generality but making calculations simpler, we will assume that system (3.1) has only one scalar Wiener process $W(t)$, and the system itself is of the form

$$dx = A(t)xdt + B(t)xdW(t) . \tag{3.4}$$

Together with system (3.1), consider the following system of linear nonhomogeneous equations:

$$dx = [A(t)x + \alpha(t)]dt + B(t)x\, dW(t) , \tag{3.5}$$

where $\alpha(t)$ is a Wiener process that is measurable and F_t-measurable for every $t \geq 0$. Here F_t is a flow of the σ-algebras from the definition of the initial system. We will assume that $\operatorname{esssup}_{t\geq 0}\mathbf{E}|\alpha(t)|^2 < \infty$. By introducing the norm $||\alpha||_2 = (\operatorname{esssup}_{t\geq 0}\mathbf{E}|\alpha(t)|^2)^{\frac{1}{2}}$, this set of random processes becomes a Banach space. Denote it by $\mathbf{B}$.

Theorem 3.1. *Let system* (3.5) *be such that for an arbitrary random process $\alpha \in \mathbf{B}$ there exists $x_0 \in \mathbf{R}^n$ such that the solution $x(t, x_0)$ is mean square bounded on the positive semiaxis. Then system* (3.4) *is mean square exponentially dichotomous on the positive semiaxis.*

Proof. Denote by $G_1 \subset \mathbf{R}^n$ the set of all initial conditions for solutions of system (3.4) such that they would be mean square bounded. Since system (3.4) is linear, it follows that G_1 is a subspace of $\mathbf{R}^n$. We will show that it is R^- in the definition of the exponential dichotomy. To prove the theorem, we will need the following lemma.

Lemma 3.1. *Let conditions of Theorem 3.1 be satisfied. Then to every random process $\alpha \in \mathbf{B}$ there corresponds a unique mean square bounded solution $x(t)$ of system* (3.5) *such that $x(0) \in G_1^{\perp} = G_2$, where $G_1^{\perp}$ denotes the orthogonal complement of G_1. This solution satisfies the estimate*

$$||x||_2 \leq K||\alpha||_2 , \tag{3.6}$$

where K is a positive constant independent of $\alpha(t)$.

Proof of Lemma 3.1. Let $\alpha(t) \in \mathbf{B}$. Then, by the conditions, there exists a mean square bounded solution $x(t, x_0)$ of system (3.5).

Denote by P_1 and P_2 the pair of complementary projections onto G_1 and G_2, respectively. Let $x_1(t)$ be a solution of equation (3.4) satisfying the initial condition $x_1(0) = P_1x_0$. By the definition of the space G_1, it follows that such a solution is mean square bounded on the semiaxis $t \geq 0$. It is clear that $x_2(t) = x(t, x_0) - x_1(t)$ is a solution of system (3.5). It follows at once that this solution is mean square bounded on the positive semiaxis. Since $x_2(0) = x_0 - P_1 x_0 = P_2 x_0 \in G_2$, this implies existence of the bounded solution mentioned in the lemma.

Its uniqueness follows, since the difference of two such solutions is a mean square bounded solution of the homogeneous equation (3.4) starting in G_2. This is possible only if this solution is zero.

Let us now prove inequality (3.6). To this end, consider a space $\mathbf{B_1}$ of all solutions, which are bounded with respect to the norm $||.||_2$, of the stochastic equation

$$x(t) = x(0) + \int_0^t (A(s)x(s) + \alpha(s))ds + \int_0^t B(s)x(s)dW(s) \tag{3.7}$$

with the condition that $x(0) \in G_2, \alpha(t) \in B$.

Equation (3.7) defines a bijective linear operator $F : \mathbf{B_1} \to \mathbf{B}$ that maps every $x \in \mathbf{B_1}$ into $\alpha \in \mathbf{B}$ such that $x(t)$ is a mean square bounded solution of equation (3.5). Indeed, if $x \in \mathbf{B_1}$, then the definition of the space $\mathbf{B_1}$ implies that there exists $\alpha \in \mathbf{B}$ such that $x(t)$ is a solution of equation (3.7) with the given $\alpha(t)$. Suppose there is another $\alpha_1 \in \mathbf{B}$ such that $x(t)$ is a solution of the equation

$$x(t) = x(0) + \int_0^t (A(s)x(s) + \alpha_1(s))ds + \int_0^t B(t)x(s)dW(s)\,. \tag{3.8}$$

By subtracting (3.8) from (3.7) we get that

$$\int_0^t (\alpha(s) - \alpha_1(s))\,ds = 0\,, \tag{3.9}$$

which can be true only if $\alpha(t) = \alpha_1(t)$ for almost all $t \geq 0$ with probability 1. This implies that α and α_1 coincide as elements of the space $\mathbf{B}$. It has already been shown that for an arbitrary $\alpha \in \mathbf{B}$ there exists a unique solution $x(t)$ of equation (3.7) such that $x(0) \in G_2,\ x \in \mathbf{B_1}$. It is immediate that the operator F is linear.

Let us introduce a norm in $\mathbf{B_1}$ by

$$|||x||| = \|x\|_2 + \|Fx\|_2\,, \tag{3.10}$$

which immediately yields continuity of the operator F with respect to $||| \cdot |||$. Let us show that the space $\mathbf{B_1}$ is complete. Let $\{x_n(t)\}$ be a Cauchy sequence.

It follows from (3.10) that it is a Cauchy sequence in $\mathbf{B}$, hence it has a limit x in $\mathbf{B}$. Since solutions of system (3.5) are continuous with probability 1 and have bounded moments, we see that $\operatorname{esssup}_{t\geq 0}\mathbf{E}|x_n(t)-x_m(t)|^2) = \sup_{t\geq 0}\mathbf{E}|x_n(t)-x_m(t)|^2$. Hence, for an arbitrary $t \geq 0$, we will have that $\mathbf{E}|x_n(t)-x(t)|^2 \to 0$ for $n \to \infty$. Hence, $|x_n(0)-x(0)| \to 0$ for $n \to \infty$. And since $x_n(0) \in G_2$ and G_2 is a subspace of $\mathbf{R}^n$, we see that $x(0) \in G_2$.

It follows from the inequality $\|F(x_n - x_m)\|_2 \leq \|F\| \, \||\, x_n - x_m \||$ that the sequence $Fx_n = \alpha_n$ is Cauchy in $\mathbf{B}$ and, hence, it has a limit $\alpha \in \mathbf{B}$ such that $\operatorname{esssup}_{t\geq 0}\mathbf{E}|\alpha_n(t)-\alpha(t)|^2 \to 0$ for $n \to \infty$.

Let us show that $x(t)$ satisfies the equation

$$x(t) = x(0) + \int_0^t (A(s)x(s)+\alpha(s))ds + \int_0^t B(t)x(s)dW(s)\,. \tag{3.11}$$

Since $A(t)$ and $B(t)$ are continuous and bounded and $x \in \mathbf{B}$, we see that $x(t)$ is a F_t-measurable process and both integrals in (3.11) exist. Let us now estimate, for $t > 0$, the difference between the left- and the right-hand sides of (3.11). We have

$$\begin{aligned}
&\mathbf{E}|x(t) - x(0) - \int_0^t (A(s)x(s)+\alpha(s))ds - \int_0^t B(t)x(s)dW(s)|^2 \\
&\quad\leq \mathbf{E}(|x(t)-x_n(t)| + |x_n(t) - x(0) - \int_0^t (A(s)x(s)+\alpha(s))ds \\
&\quad - \int_0^t B(t)x(s)dW(s)|)^2 \leq 2[\mathbf{E}|x(t)-x_n(t)|^2 + \mathbf{E}|x_n(t)-x(0) \\
&\quad - \int_0^t (A(s)x(s)+\alpha(s))ds - \int_0^t B(t)x(s)dW(s)|^2]\,.
\end{aligned} \tag{3.12}$$

The first term in the above formula approaches zero as $n \to \infty$. Let us estimate the second term. Since x_n belongs to $\mathbf{B_1}$ for every n, $x_n(t)$ satisfies the equation

$$x_n(t) = x_n(0) + \int_0^t (A(s)x_n(s)+\alpha_n(s))ds + \int_0^t B(t)x_n(s)dW(s)\,. \tag{3.13}$$

By substituting (3.13) into (3.12) we obtain that the second term in (3.12) can be estimated as follows:

$$\begin{aligned}
&3\Bigg[\mathbf{E}|x_n(0)-x(0)|^2 + \mathbf{E}\bigg(\int_0^t (||A(s)|||x_n(s)-x(s)| + |\alpha_n(s)-\alpha(s)|)ds\bigg)^2 \\
&\quad + \mathbf{E}\bigg|\int_0^t B(s)(x_n(s)-x(s))dW(s)\bigg|^2\Bigg]
\end{aligned}$$

$$\leq 3\left[\mathbf{E}|x_n(0)-x(0)|^2 + 2t\int_0^t ||A(s)||^2\mathbf{E}|x_n(s)-x(s)|^2 ds\right.$$
$$\left. + 2t\int_0^t \mathbf{E}|\alpha_n - \alpha(s)|^2 ds + \int_0^t ||B(s)||^2\mathbf{E}|x_n(s)-x(s)|^2 ds\right].$$

Each term in the latter formula approaches zero as $n \to \infty$. It then follows from (3.12) that $x(t)$ satisfies (3.11) with probability 1 for every $t \geq 0$. Hence, the space $\mathbf{B_1}$ is complete and thus the linear continuous operator F bijectively maps the Banach space $\mathbf{B_1}$ onto the Banach space $\mathbf{B}$. By the Banach theorem, the inverse operator F^{-1} is also continuous. Thus, a solution of (3.5) satisfies the estimate

$$||x||_2 \leq |||x||| \leq ||F^{-1}||||\alpha||_2\,,$$

which proves (3.4). □

Let us now continue to prove the theorem. Let $x(t)$ be a nonzero solution of system (3.4) such that $x(0) \in G_1$. Note that, since the system is linear, the point zero is unreachable for this solution. Set

$$y(t) = x(t)\int_0^t \frac{\beta(s)}{(\mathbf{E}|x(s)|^2)^{\frac{1}{2}}}\, ds\,, \tag{3.14}$$

where

$$\beta(t) = \begin{cases} 1, & 0 \leq t \leq t_0 + \tau, \\ 1-(t-t_0-\tau), & t_0+\tau \leq t \leq t_0+\tau+1, \\ 0, & t \geq t_0 + \tau + 1\,. \end{cases}$$

It is clear that $y(t)$ is F_t-measurable and the stochastic differential has the form

$$dy = \int_0^t \frac{\beta(s)}{(\mathbf{E}|x(s)|^2)^{\frac{1}{2}}}\, ds dx + x(t)\frac{\beta(t)}{(\mathbf{E}|x(t)|^2)^{\frac{1}{2}}} dt$$
$$= \int_0^t \frac{\beta(s)}{(\mathbf{E}|x(s)|^2)^{\frac{1}{2}}}\, ds(A(t)xdt + B(t)xdW(t)) + x(t)\frac{\beta(t)}{(\mathbf{E}|x(t)|^2)^{\frac{1}{2}}} dt$$
$$= A(t)ydt + x(t)\frac{\beta(t)}{(\mathbf{E}|x(t)|^2)^{\frac{1}{2}}} dt + B(t)ydW(t)\,.$$

That is, $y(t)$ is a solution of equation (3.5) with $\alpha(t) = x(t)\frac{\beta(t)}{(\mathbf{E}|x(t)|^2)^{\frac{1}{2}}}$. It is clear that $||y||_2 < \infty$ and $\alpha \in \mathbf{B}$. And since $y(0) = 0 \in G_2$, by the above lemma, we have

$$||y||_2 \leq K||\alpha||_2.$$

Whence,

$$(\mathbf{E}|y(t)|^2)^{\frac{1}{2}} \leq K(\operatorname{esssup}_{t\geq 0}\mathbf{E}|\alpha(t)|^2)^{\frac{1}{2}} \leq K$$

for $t \geq 0$. In particular, for $t = t_0 + \tau$, we get

$$(\mathbf{E}|y(t_0+\tau)|^2)^{\frac{1}{2}} = (\mathbf{E}|x(t_0+\tau)|^2)^{\frac{1}{2}} \int_0^{t_0+\tau} \frac{ds}{\mathbf{E}|x(s)|^2} \leq K\,. \tag{3.15}$$

Consider now the function

$$\psi(t) = \int_0^t \frac{1}{(\mathbf{E}|x(s)|^2)^{\frac{1}{2}}}\, ds.$$

Using (3.15) we have

$$\frac{\psi'(t_0+\tau)}{\psi^{(}t_0+\tau)} \geq \frac{1}{K}\,,$$

and integrating it from 1 to τ we get

$$\psi(t_0+\tau) \geq \psi(t_0+1)\exp\left\{\frac{\tau-1}{K}\right\} \tag{3.16}$$

for $\tau \geq 1$. Since $x(t)$ is a solution of system (3.4), we have

$$x(t) = x(t_0) + \int_{t_0}^t A(s)x(s)\,ds + \int_{t_0}^t B(s)x(s)\,dW(s) \tag{3.17}$$

and, thus, for $t \in [t_0\ t_0+1]$,

$$\mathbf{E}|x(t)|^2 \leq 3(\mathbf{E}|x(t_0)|^2 + \int_{t_0}^{t_0+1} ||A(s)||^2\mathbf{E}|x(s)|^2\,ds + \int_{t_0}^{t_0+1} ||B(s)||^2\mathbf{E}|x(s)|^2\,ds)\,.$$

The above and the Gronwall-Bellman inequality yield

$$\mathbf{E}|x(t)|^2 \leq 3\mathbf{E}|x(t_0)|^2\exp\{C\}\,, \tag{3.18}$$

where $C > 0$ is a constant independent of t_0. Thus

$$\begin{aligned}\psi(t_0+1) &= \int_0^{t_0+1} \frac{1}{(\mathbf{E}|x(s)|^2)^{\frac{1}{2}}}\,ds \\ &\geq \int_{t_0}^{t_0+1} \frac{1}{(\mathbf{E}|x(s)|^2)^{\frac{1}{2}}}\,ds \geq \frac{1}{3^{\frac{1}{2}}}(\mathbf{E}|x(t_0)|^2)^{-\frac{1}{2}}\exp\left\{-\frac{C}{2}\right\}.\end{aligned}$$

This inequality, as well as (3.15) and (3.16), show that, for $\tau \geq 1$,

$$(\mathbf{E}|x(t_0+\tau)|^2)^{\frac{1}{2}} \leq \frac{K}{\psi(t_0+\tau)} \leq N(\mathbf{E}|x(t_0)|^2)^{\frac{1}{2}}\exp\left\{-\frac{\tau}{K}\right\}, \tag{3.19}$$

where $N > 0$ is a constant independent of t_0 and τ. For $\tau \le 1$, inequality (3.18) gives

$$\mathbf{E}|x(t_0+\tau)|^2 \le 3\mathbf{E}|x(t_0)|^2 \exp\left\{\frac{2}{K} + C - \frac{2\tau}{K}\right\}. \tag{3.20}$$

Since $t_0 \ge 0$ is arbitrary, the first inequality in Definition 3.1 follows from (3.19) and (3.20) with

$$\gamma = \frac{2}{K}, \quad K_1 = \max\left\{N^2; 3\exp\left\{\frac{2}{K} + C\right\}\right\}.$$

Let us prove the second inequality in Definition 3.1. Let $x(t)$ be a nonzero solution of equation (3.4) with $x(0) \in G_2$. As above, it can be proved that

$$y(t) = x(t)\int_t^\infty \frac{\beta(s)}{(\mathbf{E}|x(s)|^2)^{\frac{1}{2}}}\,ds$$

is a solution of equation (3.5) with

$$\alpha(t) = -\frac{x(t)}{(\mathbf{E}|x(s)|^2)^{\frac{1}{2}}}\beta(t).$$

Since $y(t) = 0$ for $t \ge t_0 + \tau + 1$, we have $\sup_{t\ge 0}\mathbf{E}|y(t)|^2 < \infty$. It is clear that $y(0) \in G_2$. Hence, by the above lemma,

$$(\mathbf{E}|y(t)|^2)^{\frac{1}{2}} = (\mathbf{E}|x(t)|^2)^{\frac{1}{2}}\int_t^\infty \frac{\beta(s)}{\sqrt{\mathbf{E}|x(s)|^2}}\,ds \le K.$$

So, for arbitrary $\tau \ge 0$, $t \ge 0$, and a sufficiently large $n \in \mathbf{Z}^+$, we have

$$\int_t^n \frac{\beta(s)}{\sqrt{\mathbf{E}|x(s)|^2}}\,ds \le \int_t^\infty \frac{\beta(s)}{\sqrt{\mathbf{E}|x(s)|^2}}\,ds \le \frac{K}{\sqrt{\mathbf{E}|x(t)|^2}}. \tag{3.21}$$

The following estimate follows from (3.21) for arbitrary $\tau \ge 0$ and a sufficiently large natural n:

$$\int_t^n \frac{\beta_\tau(s)}{\sqrt{\mathbf{E}|x(s)|^2}}\,ds \le \frac{K}{\sqrt{\mathbf{E}|x(t)|^2}},$$

where $\beta_\tau(s)$ is the function $\beta(s)$ for a fixed τ. By passing to the limit in the latter inequality as $\tau \to \infty$, we obtain for every n that

$$\int_t^n \frac{1}{\sqrt{\mathbf{E}|x(s)|^2}}\,ds \le \frac{K}{\sqrt{\mathbf{E}|x(t)|^2}}.$$

The left-hand side of the above inequality is monotone and bounded in n and, hence, passing to the limit in this inequality as $n \to \infty$ we obtain

$$\int_t^\infty \frac{1}{\sqrt{\mathbf{E}|x(s)|^2}}\, ds \le \frac{K}{\sqrt{\mathbf{E}|x(t)|^2}}\,. \tag{3.22}$$

Set

$$\psi(t) = \int_t^\infty \frac{1}{\sqrt{\mathbf{E}|x(s)|^2}}\, ds.$$

Then (3.22) yields

$$\psi'(t) \le -\frac{1}{K}\psi(t),$$

which implies the inequality

$$\psi(t) \le \psi(t_0) \exp\left\{-\frac{1}{K}(t-t_0)\right\}. \tag{3.23}$$

Since $x(t)$ is a solution of system (3.4), writing a linear system of ordinary equations for its second moments, using the fact that its coefficients are bounded on the semiaxis, and applying the Gronwall-Bellman inequality we obtain for $\tau \ge t$ that

$$\mathbf{E}|x(\tau)|^2 \le C_1 \mathbf{E}|x(t)|^2 \exp\{L(\tau - t)\}\,,$$

where L and C_1 are positive constants independent of τ and t.

Hence,

$$\begin{aligned}(\mathbf{E}|x(t)|^2)^{\frac{1}{2}}\psi(t) &= (\mathbf{E}|x(t)|^2)^{\frac{1}{2}} \int_t^\infty \frac{1}{\sqrt{\mathbf{E}|x(s)|^2}}\, ds \\ &\ge \int_t^\infty \frac{1}{\sqrt{C_1}} \exp\left\{-\frac{L}{2}(s-t)\right\} ds = l\,,\end{aligned}$$

where l is a positive constant. Then it follows from (3.22) and (3.23) that

$$(\mathbf{E}|x(t)|^2)^{\frac{1}{2}} \ge \frac{l}{\psi(t)} \ge \frac{l}{\psi(t_0)} \exp\left\{\frac{1}{K}(t-t_0)\right\} \ge \frac{l}{K} \exp\left\{\frac{1}{K}(t-t_0)\right\}(\mathbf{E}|x(t_0)|^2)^{\frac{1}{2}}\,.$$

This estimate is the second inequality in the definition of exponential dichotomy. $\square$

In the theory of ordinary differential equations, together with the direct theorem on dichotomy, one also proves a converse, that the exponential dichotomy of a homogeneous system implies the existence of a bounded solution of the nonhomogeneous system and that this solution admits the representation

$$y(t) = \int_0^\infty G(t,\tau) f(\tau)\, d\tau, \tag{3.24}$$

where $G(t,\tau)$ is Green's function of the linear system,

$$G(t,\tau) = \begin{cases} \Phi(0,t)P_1(\Phi(0,\tau))^{-1}, & t \geq \tau\,, \\ -\Phi(0,t)P_2(\Phi(0,\tau))^{-1}, & t < \tau\,, \end{cases} \tag{3.25}$$

where $\Phi(\tau,t)$ is a matriciant of the homogeneous system.

For stochastic nonhomogeneous systems

$$dx = (A(t)x + \alpha(t))dt + (B(t)x + \beta(t))]dW(t) \tag{3.26}$$

one can also write the formal representation

$$y(t) = \int_0^\infty G(t,\tau)\alpha(\tau)\,d\tau + \int_0^\infty G(t,\tau)\beta(\tau)\,dW(\tau)\,, \tag{3.27}$$

however, then $y(t)$ will no longer be a F_t-measurable process. Thus a use of Green's function yields a similar result only in the case where the homogeneous system is exponentially stable and the nonhomogeneous system has the form

$$dx = [A(t)x + \alpha(t)]dt + \beta(t)dW(t)\,. \tag{3.28}$$

Theorem 3.2. *Let the homogeneous system*

$$dx = A(t)xdt \tag{3.29}$$

be exponentially stable on the positive semiaxis. Then system (3.28) *has a solution, mean square bounded on the positive semiaxis, for arbitrary* $\alpha(t), \beta(t) \in B$. *Here, all bounded solutions of system* (3.28) *admit the representations*

$$x = \psi(t) + \int_0^t \Phi(t,\tau)\alpha(\tau)\,d\tau + \int_0^t \Phi(t,\tau)\beta(\tau)\,dW(\tau)\,, \tag{3.30}$$

where $\psi(t)$ *is an arbitrary solution of system* (3.29) *and* $\Phi(t,\tau)$ *is a matriciant of system* (3.29).

Proof. Since system (3.29) is exponentially stable, its matriciant satisfies the estimate

$$||\Phi(t,\tau)|| \leq K\exp\{-\gamma(t-\tau)\} \tag{3.31}$$

for $t \geq \tau \geq 0$ with some positive K and γ. Let us show that $x(t)$ defined by (3.30) is mean square bounded for $t \geq 0$. To this end, we will prove that all the terms are bounded. Indeed, $\psi(t)$ is a deterministic function bounded on the semiaxis. Consider the second term. It follows from the Cauchy-Bunyakovskii inequality that

$$\mathbf{E}\Big|\int_0^t \Phi(t,\tau)\alpha(\tau)\,d\tau\Big|^2 \le \mathbf{E}\bigg(\int_0^t ||\Phi(t,\tau)|||\alpha(\tau)|\,d\tau\bigg)^2$$
$$\le K^2\mathbf{E}\bigg(\int_0^t \exp\left\{\frac{-\gamma(t-\tau)}{2}\right\}\exp\left\{\frac{-\gamma(t-\tau)}{2}\right\}|\alpha(\tau)|\,d\tau\bigg)^2$$
$$\le K^2\int_0^t \exp\{-\gamma(t-\tau)\}\,d\tau\int_0^t \exp\{-\gamma(t-\tau)\}\mathbf{E}|\alpha(\tau)|^2\,d\tau < C\,,$$

where $C > 0$ is a constant.

To obtain an estimate for the third term, we will us properties of a stochastic integral. We have

$$\mathbf{E}\bigg|\int_0^t \Phi(t,\tau)\beta(\tau)\,dW(\tau)|^2 \le \int_0^t ||\Phi(t,\tau)||^2\mathbf{E}|\beta(\tau)\bigg|^2\,d\tau$$
$$\le K^2\int_0^t \exp\{-2\gamma(t-\tau)\}\,d\tau\,\operatorname{esssup}_{t\ge 0}\mathbf{E}|\beta(t)|^2 < C_1\,,$$

which proves that the third term is bounded. Hence, the expression in (3.30) is mean square bounded. It is clear that $x(t)$ is F_t-measurable and, by [186, p. 234], it is a solution of system (3.28). □

3.2 A study of dichotomy in terms of quadratic forms

In the previous section, the study of exponential dichotomy of a linear stochastic Ito system was related to the existence of solutions, mean square bounded on the positive semiaxis, of a nonhomogeneous system. The obtained results are theoretical in nature and, generally speaking, are not effective for practical establishing the dichotomy. In this section, we obtain dichotomy conditions in terms of quadratic forms. These conditions are more convenient from the practical point of view since, as we have mentioned before, the methods for constructing quadratic forms that satisfy certain conditions along the system are fairly well developed for stochastic Ito type systems.

So, consider a system of linear differential stochastic Ito equations,

$$dx = A(t)xdt\ + \sum_{i=1}^{m} B_i(t)xdW_i(t)\,. \tag{3.32}$$

Choose the quadratic form to be $(S(t)x, x)$, where $S(t)$ is a symmetric matrix bounded for $t \geq 0$.

The following theorem is a generalization of the known result for systems of ordinary differential equations obtained in [106, p. 3].

Theorem 3.3. *Let there exist a symmetric continuously differentiable matrix $S(t)$, bounded for $t \geq 0$, such that*

$$S^* = \frac{dS}{dt} + A^T S + SA + \sum_{i=1}^{m} B_i^T S B_i$$

is negative definite for $t \geq 0$. Then system (3.32) *is mean square exponentially dichotomous.*

Remark. Here, the matrix $S^*(t)$ is negative definite in the sense that there exists a constant $N > 0$ such that the quadratic form $(S^*(t)x, x)$ satisfies the inequality

$$(S^*(t)x, x) \leq -N|x|^2$$

for all $t \geq 0$ and $x \in \mathbf{R}^n$.

Proof. Denote by H_τ^t a matriciant of system (3.32) ($H_\tau^\tau = E$ is the identity matrix). As it follows from [186, p. 230], such a matriciant always exists for $t \geq \tau$, has the second moment, its determinant is not zero with probability 1, and a solution $x(t, x_0)$ of system (3.32) can be written as

$$x(t, x_0) = H_\tau^t x(\tau, x_0)\,. \tag{3.33}$$

Consider the quadratic form

$$\begin{aligned}(S_t x, x) &= \mathbf{E}(S(t)x(t, x), x(t, x)) \\ &= \mathbf{E}(S(t)H_0^t x, H_0^t x) = \mathbf{E}((H_0^t)^T S(t) H_0^t x, x)\,. \end{aligned} \tag{3.34}$$

Here $x(t, x)$ is a solution of system (3.32) satisfying the initial condition $x(0, x) = x$ and x is a nonrandom vector. The expression in formula (3.34) makes sense, since the second moments of solutions of system (3.32) exist.

By expressing the difference

$$(S(t)x(t, x), x(t, x)) - (S(\tau)x(\tau, x), x(\tau, x))$$

for arbitrary $t \geq \tau \geq 0$ with a use of the Ito formula, we get

$$(S(t)x(t,x), x(t,x)) - (S(\tau)x(\tau,x), x(\tau,x))$$
$$= \int_\tau^t LV(s, x(s,x))ds + \sum_{i=1}^m \int_\tau^t \left(B_i(s)x, \frac{\partial V(s, x(s,x))}{\partial x}\right) dW_i(s)\,, \quad (3.35)$$

where $V(t, x(t,x)) = (S(t)x(t,x), x(t,x))$ and L is a generating operator for the Markov process in system (3.32). This operator, by [70, p. 109], has the form

$$LV = \frac{\partial V}{\partial t} + \left(A(t)x, \frac{\partial}{\partial x}\right)V + \frac{1}{2}\sum_{i=1}^m \left(B_i(t)x, \frac{\partial}{\partial x}\right)^2 V\,.$$

This representation and the conditions of the theorem imply that LV is a negative definite quadratic form.

Note that by a result from [70, p. 205], the point $x = 0$ is unreachable for the process $x(t,x)$ for $x \neq 0$ and, hence, by taking expectation of both sides in the above formula, we obtain using the conditions of the theorem that

$$(S_t x, x) < (S_\tau x, x) \quad (3.36)$$

for arbitrary $t > \tau \geq 0, x \neq 0$.

Let us now show that estimate (3.2) holds for the points $x \in \mathbf{R}^n$ such that $(S_t x, x) \geq 0$ for $t \geq 0$, and the points $x \in \mathbf{R}^n$ such that $(S_t x, x) \leq 0$ for $t \geq 0$ satisfy estimate (3.3).

To prove estimate (3.2), set

$$V_\varepsilon(t) = (S_t x, x) + \ \varepsilon \mathbf{E}|x(t,x)|^2,$$

assuming that ε is a sufficiently small constant. It follows from (3.35) that

$$\mathbf{E}(S(t)x(t,x), x(t,x)) - \mathbf{E}(S(\tau)x(\tau,x), x(\tau,x)) = \int_\tau^t \mathbf{E}LV(s, x(s,x))ds.$$

By differentiating it with respect to t, we get

$$\frac{d}{dt}(S_t x, x) = \mathbf{E}L(S(t)x(t,x), x(t,x)) \leq -N\mathbf{E}|x(t,x)|^2, \quad (3.37)$$

where N is a positive constant.

It is clear that

$$L|x|^2 = (C(t)x, x),$$

where

$$C(t) = A(t) + A^T(t) + \sum_{j=1}^{m} (B_j(t))^T B_j(t).$$

Since $A(t)$ and $B_j(t)$ are bounded,

$$|L|x|^2| = |(C(t)x, x)| \leq D|x|^2, \tag{3.38}$$

where $D = \sup_{t\geq 0} ||C(t)||$.

Hence,

$$V_\varepsilon(t) - V_\varepsilon(\tau) = \int_\tau^t (\mathbf{E}LV(s, x(s,x)) + \varepsilon \mathbf{E}L|x(s,x)|^2) ds.$$

This shows that

$$\frac{dV_\varepsilon}{dt} \leq -(N - \varepsilon D)\mathbf{E}|x(t,x)|^2 = -N_1 \mathbf{E}|x(t,x)|^2. \tag{3.39}$$

Since

$$V_\varepsilon(t) \leq (C_1 + \varepsilon)\mathbf{E}|x(t,x)|^2, \tag{3.40}$$

where $C_1 = \sup_{t\geq 0} ||S(t)||$, and

$$\mathbf{E}|x(t,x)|^2 \leq \frac{(S_t x, x) + \varepsilon \mathbf{E}|x(t,x)|^2}{\varepsilon} = \frac{V_\varepsilon(t)}{\varepsilon}, \tag{3.41}$$

inequality (3.39) yields the estimate

$$\frac{dV_\varepsilon(t)}{dt} \leq -N_1 \mathbf{E}|x(t,x)|^2 \leq \frac{-N_1}{C_1 + \varepsilon} V_\varepsilon(t) = -\gamma V_\varepsilon(t).$$

By integrating this inequality over the segment $[\tau,\ t]$, we get

$$V_\varepsilon(t) \leq V_\varepsilon(\tau) \exp\{-\gamma(t-\tau)\}$$

for $t \geq \tau$. Or, using (3.40) and (3.41) we get

$$\begin{aligned} \mathbf{E}|x(t,x)|^2 &\leq \frac{V_\varepsilon(t)}{\varepsilon} \leq \frac{V_\varepsilon(\tau)}{\varepsilon} \exp\{-\gamma(t-\tau)\} \\ &\leq \left(1 + \frac{C_1}{\varepsilon}\right) \exp\{-\gamma(t-\tau)\} \mathbf{E}|x(\tau,x)|^2 . \end{aligned}$$

The latter inequality proves (3.2) for $K = 1 + \frac{C_1}{\varepsilon}$.

Let now $x \in \mathbf{R}^n$ be such that $(S_t x, x) \le 0$ for $t \ge 0$. We show that inequality (3.3) holds for such x.

To this end, consider the function

$$V_\varepsilon^1(t) = -(S_t x, x) + \varepsilon \mathbf{E}|x(t,x)|^2 \,,$$

where again ε is a sufficiently small positive number.

By the conditions of the theorem, the quadratic form $-L(S(t)x, x)$ is positive definite and, as before, we get the estimate

$$\begin{aligned}\frac{dV_\varepsilon^1(t)}{dt} &\ge N\mathbf{E}|x(t,x)|^2 + \varepsilon \mathbf{E}(C(t)x(t,x), x(t,x)) \\ &\ge (N - \varepsilon D)\mathbf{E}|x(t,x)|^2 \ge \frac{N_1}{C_1 + \varepsilon} V_\varepsilon^1(t) = \gamma V_\varepsilon^1(t) \,.\end{aligned}$$

Integrating this inequality we obtain

$$V_\varepsilon^1(t) \ge V_\varepsilon^1(\tau) \exp\{\gamma(t-\tau)\}$$

for $t \ge \tau$. This, together with the inequalities

$$V_\varepsilon^1(t) \ge \varepsilon \mathbf{E}|x(t,x)|^2$$

and (3.40) gives the needed inequality (3.3),

$$\begin{aligned}\mathbf{E}|x(t,x)|^2 &\ge \frac{V_\varepsilon^1(t)}{C_1+\varepsilon} \ge \frac{V_\varepsilon^1(\tau)}{C_1+\varepsilon} \exp\{\gamma(t-\tau)\} \\ &\ge \frac{\varepsilon \mathbf{E}|x(\tau,x)|^2}{C_1+\varepsilon} \exp\{\gamma(t-\tau)\} = \frac{1}{\frac{C_1}{\varepsilon}+1} \mathbf{E}|x(\tau,x)|^2 \exp\{\gamma(t-\tau)\} \\ &= K_1 \exp\{\gamma(t-\tau)\} \mathbf{E}|x(\tau,x)|^2\end{aligned}$$

for $t \ge \tau \ge 0$.

Let us show that the space $\mathbf{R}^n$ can be decomposed into a direct sum of subspaces R^- and R^+. We take R^- to be the set of all initial conditions $x \in \mathbf{R}^n$ for solutions of system (3.32) such that $\mathbf{E}|x(t,x)|^2$ is bounded for $t \ge 0$. Using representation (3.33) for a solution of system (3.32), it is easy to see that this set is a linear subspace of $\mathbf{R}^n$. For all points of this subspace, we have that $(S_t x, x) \ge 0$. Indeed, if not there would exist a point $t_0 > 0$ such that $(S_{t_0} x, x) < 0$. Then inequality (3.36) would yield the estimate

$$(S_t x, x) < 0 \tag{3.42}$$

for $t \geq t_0$. Which, using the above, would give

$$\mathbf{E}|x(t,x)|^2 \geq K_1 \exp\{\gamma(t-t_0)\}\mathbf{E}|x(t_0,x)|^2\,,$$

which holds for all $t \geq t_0$ which is a contradiction, since solutions that start in R^- are bounded on the semiaxis.

Hence, for arbitrary $x \in R^-$, we have the inequality $(S_t x, x) \geq 0$ for $t \geq 0$. This shows that estimate (3.2) holds for $x \in R^-$.

Set R^+ to be $R^+ = (R^-)^\perp$, the orthogonal complement to R^-, and show that, for all $x \in R^+$, the inequality $(S_t x, x) \leq 0$ holds for $t \geq t(x)$. Indeed, otherwise, the expression $(S_t x, x)$ would be positive for all $t \geq 0$ for a nonzero $x \in R^+$. This would lead to estimate (3.2) for the solution $x(t,x)$, $x(0,x) = x \in \mathbf{R}^+$, which implies that $\mathbf{E}|x(t,x)|^2$ is bounded. The latter means that $x \in R^-$. But the subspaces R^- and R^+ intersect only in zero vector. This contradiction shows that, for every $x \in R^+$ there is a finite time $t(x)$ such that $(S_t x, x) \leq 0$ for $t \geq t(x)$. Hence, as we have shown above, if $x \in R^+$, estimate (3.3) holds for $t \geq \tau \geq t(x)$. Let us show that it holds for $0 \leq \tau \leq t \leq t(x)$.

To this end, let us first prove that it is possible to chose the time $t(x)$ to be the same for all $x \in \mathbf{R}^+$.

Suppose that this is not true. Then there is a sequence of real numbers $t_n \to \infty$ and a sequence $x_n \in R^+$ such that $(S_{t_n} x_n, x_n) > 0$. Consider the sequence $y_n = \frac{x_n}{|x_n|}$. It is clear that $(S_{t_n} y_n, y_n) > 0$. Since R^+ is a subspace, it follows that $y_n \in R^+$ for arbitrary natural n.

Choose a convergent subsequence of y_n. Without loss of generality, we can assume that y_n is itself convergent. Denote $y_0 = \lim_{n\to\infty} y_n$. Since the subspace R^+ is closed, it follows that $y_0 \in R^+$. Then there is a finite time $T > 0$ for y_0 such that $(S_t y_0, y_0) \leq 0$ for $t \geq T$ and, hence, the solution $x(t, y_0)$ of system (3.32), for $t \geq T$, admits the estimate

$$\mathbf{E}|x(t,y_0)|^2 \geq K_1 \exp\{\gamma(t-T)\}\mathbf{E}|x(T,y_0)|^2\,. \tag{3.43}$$

Choose t_1 such that

$$K_1 \exp\{\gamma(t_1 - T)\} = 2\,.$$

Since the mean square is continuous with respect to the initial conditions and $y_n \to y_0$ for $n \to \infty$, it follows that for arbitrary $\varepsilon > 0$ in the segment $[0,\ t_1]$ there exists a number p such that, for $n \geq p$, we have

$$\sup_{t\in[0,\ t_1]} \mathbf{E}|x(t,y_0) - x(t,y_n)|^2 < \varepsilon\,. \tag{3.44}$$

Assume that p is so large that $t_n > t_1$ for $n \geq p$. This means that $(S_{t_1}y_n, y_n) > 0$ for $n \geq p$ and, hence, the solution $x(t, y_n)$ satisfies the following estimate for $T \leq t \leq t_1$:

$$\mathbf{E}|x(t, y_n)|^2 \leq K \exp\{-\gamma(t-T)\}\mathbf{E}|x(T, y_n)|^2 . \tag{3.45}$$

Note that $K_1 = \frac{1}{K}$.

By introducing the norm

$$||x(t, y_0)||_2 = (\mathbf{E}|x(t, y_0)|^2)^{\frac{1}{2}} ,$$

we get from (3.44) that

$$||x(t_1, y_0) - x(t_1, y_n)||_2 < \varepsilon^{\frac{1}{2}} . \tag{3.46}$$

Now we have

$$\begin{aligned} ||x(t_1, y_0) - x(t_1, y_n)||_2 &\geq ||x(t_1, y_0)||_2 - ||x(t_1, y_n)||_2 \\ &\geq K_1^{\frac{1}{2}} \exp\{\frac{\gamma}{2}(t_1 - T)\}||x(T, y_0)||_2 \\ &\quad - K^{\frac{1}{2}} \exp\{-\frac{\gamma}{2}(t_1 - T)\}||x(T, y_n)||_2 \\ &\geq 2^{\frac{1}{2}}||x(T, y_0)||_2 - \frac{1}{2^{\frac{1}{2}}}||x(T, y_n)||_2 \\ &\geq 2^{\frac{1}{2}}||x(T, y_0)||_2 - \frac{1}{2^{\frac{1}{2}}}||x(T, y_0)||_2 - \frac{\varepsilon^{\frac{1}{2}}}{2^{\frac{1}{2}}} , \end{aligned}$$

which contradicts inequality (3.46). This means that there is a finite $T_0 > 0$ such that, if $t \geq T_0$, then

$$(S_t x, x) \leq 0 \tag{3.47}$$

for $x \in R^+$, which implies that (3.3) is true for $\tau \geq T_0$.

Let us prove that the above inequality holds for all $\tau \geq 0$. Indeed, as was shown above, an arbitrary solution of system (3.32) can be written as $x(t, x_0) = H_0^t x_0$, where H_0^t is a matriciant of system (3.32), nondegenerate with probability 1 for all $t \geq 0$. Hence the inverse matrix $(H_0^t)^{-1}$ exists and is continuous with probability 1. This and the inequality

$$\frac{1}{||(H_0^t)^{-1}||} \leq ||H_0^t)||$$

yield the existence of $\mathbf{E}\frac{1}{||(H_0^t)^{-1}||^2}$ for $t \in [0,\ T_0]$ and, hence, we have

$$\mathbf{E}\frac{1}{||(H_0^t)^{-1}||^2} \geq A$$

for $t \in [0,\ T_0]$. Then

$$|x(t,x_0)| = |H_0^t x_0| = \frac{||(H_0^t)^{-1}|||H_0^t x_0|}{||(H_0^t)^{-1}||} \geq \frac{|x_0|}{||(H_0^t)^{-1}||}$$

and, hence,

$$\mathbf{E}|x(t,x_0)|^2 \geq A|x_0|^2,$$

which implies that

$$\mathbf{E}|x(t,x_0)|^2 \geq A|x_0|^2 \frac{\exp\{\gamma t\}}{\exp\{\gamma t\}} \geq A|x_0|^2 \frac{\exp\{\gamma t\}}{\exp\{\gamma T_0\}} = B|x_0|^2 \exp\{\gamma t\} \tag{3.48}$$

for $t \in [0,\ T_0]$.

Since system (3.32) is linear, using the Gronwall-Bellman inequality we can obtain the following estimate that holds for $\tau \geq 0$:

$$\mathbf{E}|x(\tau,x_0)|^2 \leq A_1 \exp\{\alpha\tau\}|x_0|^2,$$

where $\alpha > 0,\ A_1 > 0$ are constants independent of $\tau,\ x_0$. Now, using inequality (3.48) it is easy to obtain the needed inequality (3.3) for $\tau \leq T_0$. Since it holds for $\tau \geq T_0$, there is a constant $K_2 > 0$, independent of $\tau,\ x_0$, such that, for all $t \geq \tau \geq 0$, we have

$$\mathbf{E}|x(t,x_0)|^2 \geq K_2 \exp\{\gamma_1(t-\tau)\}\mathbf{E}|x(\tau,x_0)|^2,$$

which proves the theorem. □

This theorem shows that the solutions of the system, which start in R^-, decrease in the mean square at an exponential decay rate, and the solution that start in R^+ exponentially increase in the mean square. Let us look at the behaviour of solutions $x(t,x_0)$ such that $x_0 \notin R^- \cup R^+$.

Corollary. *If $x_0 \notin R^- \cup R^+$, then the solution $x(t,x_0)$ of system* (3.32) *satisfies the condition*

$$\lim_{t\to\infty} \mathbf{E}|x(t,x_0)|^2 = \infty. \tag{3.49}$$

Proof. Let $x_0 \notin R^- \cup R^+$. Since the space $\mathbf{R}^n$ decomposes into the direct sum of R^- and R^+, we get the representation $x_0 = x_0' + x_0''$, where $x_0' \in R^-$ and $x_0'' \in R^+$, hence we have that

$$x(t,x_0) = H_0^t x_0 = H_0^t x_0' + H_0^t x_0''.$$

Thus,

$$\begin{aligned}||x(t,x_0)||_2 &\geq ||H_0^t x_0''||_2 - ||H_0^t x_0'||_2\\ &\geq K_2^{\frac{1}{2}} \exp\left\{\frac{\gamma}{2}t\right\}|x_0''| - K^{\frac{1}{2}} \exp\left\{-\frac{\gamma}{2}t\right\}|x_0'|,\end{aligned}$$

which proves the corollary. □

Using a method proposed in [81], we can describe not only the behaviour of the second moments of the solutions of (3.32), which start in R^-, but also the behaviour of the trajectories of these solutions.

Theorem 3.4. *Let system* (3.32) *be exponentially dichotomous in the mean square. Then the solutions $x(t,x_0)$ that satisfy $x(0,x_0) = x_0 \in R^-$ with probability* 1 *admit the following estimate with probability* 1*:*

$$|x(t,x_0)| < Q(\omega)\exp\{-\alpha t\}|x_0| \tag{3.50}$$

for some $\alpha > 0$, and the random variable $Q(\omega)$ is finite with probability 1.

Proof. Let $x(t,x_0)$ be a solution of system (3.32), with the initial conditions being in the subspace R^-. Then it satisfies the estimate

$$\mathbf{E}|x(t,x_0)|^2 \leq K\exp\{-\gamma t\}|x_0|^2\,. \tag{3.51}$$

For arbitrary natural $k < k_1$ and $\varepsilon_k > 0$, we have

$$\begin{aligned}\mathbf{P}\left\{\sup_{t\in[k,k_1]}|x(t,x_0)| \geq \varepsilon_k\right\} &\leq \mathbf{P}\left\{\sup_{t\in[k,\,k_1]}|x(t,x_0) - x(k,x_0)| \geq \frac{\varepsilon_k}{2}\right\}\\ &\quad + \mathbf{P}\left\{|x(k,x_0)| \geq \frac{\varepsilon_k}{2}\right\}.\end{aligned} \tag{3.52}$$

Let us estimate each term in (3.52). It follows from the Chebyshev inequality and (3.51) that

$$\mathbf{P}\left\{|x(k,x_0)| \geq \frac{\varepsilon_k}{2}\right\} \leq \frac{4}{\varepsilon_k^2}\mathbf{E}|x(k,x_0)|^2 \leq \frac{4}{\varepsilon_k^2}K\exp\{-\gamma k\}|x_0|^2\,. \tag{3.53}$$

We estimate the first term in (3.52) using properties of a stochastic Ito integral. Since

$$x(t,x_0) = x(k,x_0) + \int_k^t A(s)x(s,x_0)ds + \sum_{i=1}^m \int_k^t B_i(s)x(s,x_0)dW_i(s),$$

we see that

$$\mathbf{P}\left\{\sup_{t\in[k,\,k_1]}|x(t,x_0)-x(k,x_0)|\geq\frac{\varepsilon_k}{2}\right\}\leq\mathbf{P}\left\{\sup_{t\in[k,\,k_1]}\left|\int\limits_k^t A(s)x(s,x_0)ds\right|\geq\frac{\varepsilon_k}{4}\right\}$$

$$+\,\mathbf{P}\left\{\sup_{t\in[k,\,k_1]}\left|\sum_{i=1}^m\int\limits_k^t B_i(s)x(s,x_0)dW_i(s)\right|\geq\frac{\varepsilon_k}{4}\right\}. \tag{3.54}$$

Since the matrices $A(t), B_i(t)$ are bounded on the semiaxis by the condition, there exists $A_1>0$ such that

$$\sup_{t\geq 0}||A(t)||+\sum_{i=1}^m\sup_{t\geq 0}||B_i(t)||\leq A_1 \tag{3.55}$$

and, hence,

$$\mathbf{P}\left\{\sup_{t\in[k,\,k_1]}\left|\int\limits_k^t A(s)x(s,x_0)ds\right|\geq\frac{\varepsilon_k}{4}\right\}\leq\mathbf{P}\left\{A_1\int\limits_k^{k_1}|x(s,x_0)|ds\geq\frac{\varepsilon_k}{4}\right\}$$

$$\leq\frac{4A_1}{\varepsilon_k}\int\limits_k^{k_1}\mathbf{E}|x(s,x_0)|ds\,. \tag{3.56}$$

Applying properties of a stochastic integral to the second term in (3.54) we obtain the estimate

$$\mathbf{P}\left\{\sup_{t\in[k,\,k_1]}\left|\sum_{i=1}^m\int\limits_k^t B_i(s)x(s,x_0)dW_i(s)\right|\geq\frac{\varepsilon_k}{4}\right\}$$

$$\leq\mathbf{P}\left\{\sum_{i=1}^m\sup_{t\in[k,\,k_1]}\left|\int\limits_k^t B_i(s)x(s,x_0)dW_i(s)\right|\geq\frac{\varepsilon_k}{4}\right\}$$

$$\leq\sum_{i=1}^m\mathbf{P}\left\{\sup_{t\in[k,\,k_1]}\left|\int\limits_k^t B_i(s)x(s,x_0)dW_i(s)\right|\geq\frac{\varepsilon_k}{4m}\right\}$$

$$\leq\sum_{i=1}^m\sum_{j=1}^n\mathbf{P}\left\{\sup_{t\in[k,\,k_1]}\left|\int\limits_k^t(b_{j1}^i(s)x_1(s,x_0)+\cdots+b_{jn}^i(s)x_n(s,x_0))dW_i(s)\right|\geq\frac{\varepsilon_k}{4nm}\right\}$$

$$\leq \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{16n^2m^2}{\varepsilon_k^2}\int_k^{k_1}\mathbf{E}(b_{j1}^i(s)x_1(s,x_0)+\cdots++b_{jn}^i(s)x_n(s,x_0))^2ds$$

$$\leq \frac{A_2A_1^2}{\varepsilon_k^2}\int_k^{k_1}\mathbf{E}|x(s,x_0)|^2ds\,. \tag{3.57}$$

Here the upper index points at an element of the i-th matrix $B_i(s)$ and A_2 is a constant that depends only on n, the dimension of the space, and m, the number of the processes $W_i(t)$.

It follows from (3.56) and (3.57), with a use of (3.51) and (3.53), that

$$\mathbf{P}\left\{\sup_{t\in[k,\,k_1]}|x(t,x_0)|\geq\varepsilon_k\right\}\leq\frac{4A_1}{\varepsilon_k}\int_k^{k_1}\mathbf{E}|x(s,x_0)|ds$$
$$+\frac{A_2A_1^2}{\varepsilon_k^2}\int_k^{k_1}\mathbf{E}|x(s,x_0)|^2ds+\frac{4}{\varepsilon_k^2}K\exp\{-\gamma k\}|x_0|^2$$
$$\leq\frac{4A_1}{\varepsilon_k}\int_k^{k_1}K^{\frac{1}{2}}\exp\left\{\frac{-\gamma s}{2}\right\}|x_0|ds+\frac{A_2A_1^2}{\varepsilon_k^2}\int_k^{k_1}K\exp\{-\gamma s\}|x_0|^2ds$$
$$+\frac{4}{\varepsilon_k^2}K\exp\{-\gamma k\}|x_0|^2\,.$$

Let us put

$$\varepsilon_k=K\exp\left\{\frac{-\gamma k}{4}\right\}|x_0|$$

in the last formula and pass to the limit as $k_1\to\infty$. Then, since the sequence of the sets

$$\left\{\sup_{t\in[k,\,k_1]}|x(t,x_0)|\geq\varepsilon_k\right\}$$

is increasing, we get the estimate

$$\mathbf{P}\left\{\sup_{t\geq k}|x(t,x_0)|\geq K\exp\left\{\frac{-\gamma k}{4}\right\}|x_0|\right\}\leq\frac{4A_1K^{\frac{1}{2}}\exp\{\frac{-\gamma k}{2}\}}{K\exp\{\frac{-\gamma k}{4}\}|x_0|\frac{\gamma}{2}}$$
$$+\frac{A_2\exp\{-\gamma k\}|x_0|^2A_1^2}{K^2\exp\{\frac{-\gamma k}{2}\}|x_0|^2\gamma}+\frac{4}{K}\exp\left\{\frac{-\gamma k}{2}\right\}. \tag{3.58}$$

The quantity in the right-hand side of (3.58) is the k-th term of a convergent series, so by the Borel–Cantelli lemma there exists a finite number $N(\omega)$ such that, if $k > N(\omega)$ with probability 1, then

$$\sup_{t\geq k} |x(t, x_0)| \leq K \exp\left\{\frac{-\gamma k}{4}\right\}|x_0|\,,$$

which implies that

$$|x(t, x_0)| < K \exp\left\{\frac{-\gamma(t-1)}{4}\right\}|x_0|\,, \tag{3.59}$$

for $t > N(\omega)$ with probability 1. Since $x_0 \in R^-$ there exist linearly independent vectors $x_0^1, \dots, x_0^r$ in R^-, where r is the dimension of the subspace, such that

$$x_0 = \sum_{i=1}^{r} \alpha_i x_0^i.$$

Since a strong solution is unique and system (3.32) is linear, it follows that

$$x(t, x_0) = \sum_{i=1}^{r} \alpha_i x_i(t, x_0^i), \tag{3.60}$$

where $x_i(0, x_0^i) = x_0^i$. Every solution $x_i(t, x_0^i)$ satisfies estimate (3.59) for $t \geq N(\omega, x_0^i)$. Thus this estimate holds true for all $x_i(t, x_0^i)$ for $t \geq N_0(\omega) = \max\{N(\omega, x_0^1), \dots N(\omega, x_0^r)\}$. Using the representation $x_i(t, x_0^i) = H_0^t x_0^i$ and since the matriciant H_0^t is continuous with probability 1 we see that (3.50) holds for the linearly independent solutions $x_i(t, x_0^i)$. Because a matriciant of system (3.32) is bounded on the segment $[0,\ N_0(\omega)]$ with probability 1 it follows that $Q(\omega)$ in (3.50) is finite with probability 1. Now it is easy to obtain estimate (3.50) for arbitrary $x_0 \in R^-$ by using (3.60). $\square$

Consider now the linear deterministic system

$$\frac{dx}{dt} = A(t)x\,, \tag{3.61}$$

with the matrix $A(t)$ being bounded on the semiaxis. Suppose that this system is exponentially dichotomous for $t \geq 0$. Then, as follows from [95], there is a symmetric smooth matrix $S(t)$, bounded for $t \geq 0$, such that the quadratic form

$$\left(\left(\frac{dS}{dt}(t) + A^T(t)S(t) + S(t)A(t)\right)x, x\right) \tag{3.62}$$

is negative definite for $t \geq 0$. Let us use this matrix to study exponential mean square dichotomy of the stochastic Ito system with a small parameter,

$$dx = A(t)xdt \ + \mu \sum_{i=1}^{m} B_i(t)xdW_i(t)\,, \tag{3.63}$$

where the matrices $B_i(t)$ are bounded on the positive semiaxis. Since the form (3.62) is negative definite, it is clear that, for small enough μ, the form

$$\left(\left(\frac{dS}{dt}(t) + A^T(t)S(t) + S(t)A(t)\right) + \mu \sum_{i=1}^{m} B_i^T(t)S(t)B_i(t))x, x\right)$$

is also negative definite. Hence, using the quadratic form $(S(t)x, x)$ for system (3.63) and applying Theorem 3.3 we get the following result.

Theorem 3.5. *If the deterministic system* (3.61) *is exponentially dichotomous on the semiaxis $t \geq 0$, then there exists $\mu_0 > 0$ such that, for $\mu \leq \mu_0$, the stochastic Ito system* (3.63) *is mean square exponentially dichotomous for $t \geq 0$.*

Remark. System (3.63) can be considered as a perturbation of the deterministic system with "white noise" type random forces. Theorem 3.5 asserts that the dichotomy of the system is preserved. On the other hand, Theorem 3.5 permits to reduce the dichotomy study of an Ito system to that of a deterministic differential system.

3.3 Linear system solutions that are mean square bounded on the semiaxis

Let (Ω, F, P) be a complete probability space. Let a standard m-dimensional Wiener process $\{W(t) : t \in \mathbf{R}\}$ be defined on (Ω, F, P) such that the one-dimensional components $W_i(t), i = \overline{1, m}$, are totally independent scalar Wiener processes on the axis.

For each $t \in \mathbf{R}$, define a σ-algebra F_t to be a minimal σ-algebra generated by the sets

$$\{W(s_2) - W(s_1) : s_1 \leq s_2 \leq t\}\,.$$

Then $W(t)$ is measurable with respect to the flow F_t, and $W(t) - W(s)$ is independent of F_s for $s < t$.

Consider a stochastic differential equation,

$$dx = f(t, x)dt + g(t, x)dW(t)\,, \tag{3.64}$$

where $t \in \mathbf{R},\ x \in \mathbf{R}^n$, and the function f and the $n \times m$-dimensional matrix g satisfy conditions for existence and uniqueness of a solution of the Cauchy problem.

By definition, a solution of equation (3.64) on $\mathbf{R}$ is a random n-dimensional process $x(t)$ such that the following holds:

1) $x(t)$ is F_t-measurable for arbitrary $t \in \mathbf{R}$;

2) $x(t)$ has trajectories that are continuous with probability 1;

3) for arbitrary $-\infty < t_0 < t_1 < \infty$,

$$\sup_{t_0 \le t \le t_1} \mathbf{E}|x(t)|^2 < \infty\,;$$

4) for arbitrary $-\infty < t_0 < t_1 < \infty$ the following identity holds with probability 1:

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\, ds + \int_{t_0}^{t} g(s, x(s))\, dW(s)\,. \tag{3.65}$$

Here the first integral is usual, and the second one is an Ito integral.

We will be interested in finding conditions that would imply existence of solutions of (3.64), mean square bounded on $\mathbf{R}$, and solutions that are periodic (stationary) in the case where the equation is linear or weakly nonlinear. Here, without loss of generality, we will assume that the process $W(t)$ is one-dimensional, since the reasoning is similar in the many dimensional case.

In this section we will be considering the case where the matrix of the linear part is variable, and the conditions of boundedness and periodicity of solutions are given in terms of the system under consideration, which makes them convenient for applications.

In what follows, we will need a result that permits to differentiate a stochastic Ito integral with respect to the parameter. To make calculations less cumbersome, we only treat the one-dimensional case. The many dimensional version of the result holds true with an obvious reformulation.

Lemma 3.2. *Let the function $h(t, s)$ and its partial derivative $h'_t(t, s)$ be continuous in the totality of the variables $t, s \in \mathbf{R}$, and the random process $f(t)$ be F_t-measurable such that*

$$\int_{-\infty}^{\infty} |h(t, s)|^2 \mathbf{E}|f(s)|^2\, ds < \infty \tag{3.66}$$

for arbitrary $t \in \mathbf{R}$. If the integral

$$\int_{-\infty}^{t} |h_t'(t,s)|^2 \mathbf{E}|f(s)|^2 \, ds \tag{3.67}$$

converges uniformly with respect to t on an arbitrary line segment $[t_1,\, t_2]$, then the random process

$$y(t) = \int_{-\infty}^{t} h(t,s) f(s) \, dW(s) \tag{3.68}$$

has the stochastic differential,

$$dy(t) = \left(\int_{-\infty}^{t} h_t'(t,s) f(s) \, dW(s) \right) dt + h(t,t) f(t) dW(t) \, . \tag{3.69}$$

Remark 1. The integral in (3.68) is understood as the mean value limit of the sequence of processes,

$$y_n(t) = \int_{-n}^{t} h(t,s) f(s) \, dW(s) \, ,$$

which exists by (3.66). Such integrals satisfy the usual properties of stochastic integrals with trivial reformulations, see [41] or [93].

Remark 2. For a bounded interval of the real axis, this result was obtained in [186, p. 264].

Proof of Lemma 3.2. To prove the lemma, it is sufficient to show that

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} \left(\int_{-\infty}^{t} h_t'(t,s) f(s) \, dW(s) \right) dt + \int_{t_1}^{t_2} h(t,t) f(t) dW(t) \tag{3.70}$$

with probability 1 for arbitrary $t_1 < t_2 \le t$.

Take an arbitrary $n > 0$, and consider the proper stochastic integral

$$y_n(t) = \int_{-n}^{t} h(t,s) f(s) \, dW(s) \, . \tag{3.71}$$

Then, for arbitrary $t > -n$, we have

$$\mathbf{E}|y_n(t) - y(t)|^2 \to 0, \;\; n \to \infty \, . \tag{3.72}$$

It follows from [186, p. 264] that the following formula holds for the sequence of the random processes $y_n(t)$ and arbitrary $t_1 < t_2$ in the line segment $[-n,\, t]$:

$$y_n(t_2) - y_n(t_1) = \int_{t_1}^{t_2} \left(\int_{-n}^{t} h_t'(t,s) f(s) \, dW(s) \right) dt + \int_{t_1}^{t_2} h(t,t) f(t) \, dW(t). \tag{3.73}$$

We have

$$
\begin{aligned}
&\mathbf{E}|y(t_2)-y(t_1)-\int_{t_1}^{t_2}\Big(\int_{-\infty}^{t} h'_t(t,s)f(s)\,dW(s)\Big)\,dt\\
&\quad-\int_{t_1}^{t_2} h(t,t)f(t)\,dW(t)|^2=\mathbf{E}|y(t_2)-y_n(t_2)-y(t_1)+y_n(t_1)\\
&\quad-\int_{t_1}^{t_2}\Big(\int_{-\infty}^{t} h'_t(t,s)f(s)\,dW(s)\Big)\,dt-\int_{t_1}^{t_2} h(t,t)f(t)\,dW(t)\\
&\quad+\int_{t_1}^{t_2}\Big(\int_{-n}^{t} h'_t(t,s)f(s)\,dW(s)\Big)\,dt+\int_{t_1}^{t_2} h(t,t)f(t)\,dW(t)|^2\\
&\le 3\Bigg[\mathbf{E}|y(t_2)-y_n(t_2)|^2+\mathbf{E}|y(t_1)-y_n(t_1)|^2\\
&\quad+\mathbf{E}\Big|\int_{t_1}^{t_2}\Big(\int_{-\infty}^{t} h'_t(t,s)f(s)dW(s)\Big)dt-\int_{t_1}^{t_2}\Big(\int_{-n}^{t} h'_t(t,s)f(s)dW(s)\Big)dt\Big|^2\Bigg].
\end{aligned}
\tag{3.74}
$$

The first two terms in the latter inequality (3.74) tend to zero as $n\to\infty$ by (3.72).

Let us estimate the last term in the inequality:

$$
\begin{aligned}
&\mathbf{E}\Big|\int_{t_1}^{t_2}\Big(\int_{-\infty}^{t} h'_t(t,s)f(s)\,dW(s)\Big)dt-\int_{t_1}^{t_2}\Big(\int_{-n}^{t} h'_t(t,s)f(s)\,dW(s)\Big)dt\Big|^2\\
&\quad=\mathbf{E}\Big|\int_{t_1}^{t_2}\Big(\int_{-\infty}^{-n} h'_t(t,s)f(s)\,dW(s)\Big)\,dt\Big|^2\\
&\quad\le\mathbf{E}\Big(\int_{t_1}^{t_2}\Big|\int_{-\infty}^{-n} h'_t(t,s)f(s)\,dW(s)\Big|dt\Big)^2\\
&\quad\le(t_2-t_1)\int_{t_1}^{t_2}\mathbf{E}\Big|\int_{-\infty}^{-n} h'_t(t,s)f(s)\,dW(s)\Big|^2\,dt\\
&\quad=(t_2-t_1)\int_{t_1}^{t_2}\Big(\int_{-\infty}^{-n}(h')^2_t(t,s)\mathbf{E}|f(s)|^2\Big)\,ds\,dt\\
&\quad\le(t_2-t_1)^2\sup_{t\in[t_1,\,t_2]}\int_{-\infty}^{-n}(h')^2_t(t,s)\mathbf{E}|f(s)|^2\,ds\to 0
\end{aligned}
$$

as $n\to\infty$. This relation follows, since integral (3.67) converges uniformly in $t\in[t_1,\,t_2]$. Hence, the right-hand side of (3.74) tends to zero as $n\to\infty$. This shows that (3.70) holds, which finishes the proof. $\square$

Consider now the linear nonhomogeneous system of stochastic Ito equations for $t \in \mathbf{R}$,

$$dx = [A(t)x + \alpha(t)]dt + \beta(t)dW(t)\,, \tag{3.75}$$

where $A(t)$ is a matrix, continuous and bounded on $\mathbf{R}$, $\alpha(t)$, $\beta(t)$ are F_t-measurable random processes, continuous for each t and such that

$$\sup_{t\in\mathbf{R}} \mathbf{E}|\alpha(t)|^2 < \infty,\ \sup_{t\in\mathbf{R}} \mathbf{E}|\beta(t)|^2 < \infty. \tag{3.76}$$

Denote the class of such processes by B. Then, it is known [186, p. 234] that a solution of the Cauchy problem $x(t_0) = x_0$ for (3.75), where x_0 is F_{t_0}-measurable and has bounded second moment, exists, is unique for $t \geq t_0$, and has finite second moment for every $t \geq t_0$.

Let us find conditions for equation (3.75) to have solutions that are mean square bounded on the whole axis.

Theorem 3.6. *Let the deterministic system*

$$\frac{dx}{dt} = A(t)x \tag{3.77}$$

be exponentially stable.

Then system (3.75) *has a unique solution, it is mean square bounded on the axis for arbitrary* $\alpha(t)$, $\beta(t) \in B$, *and this solution is mean square exponentially stable.*

Proof. Exponential stability of the zero solution of system (3.77) means that its matriciant satisfies the estimate

$$||\Phi(t,s)|| \leq K \exp\{-\gamma(t-s)\} \tag{3.78}$$

for $t \geq s$ with positive constants K and γ.

Denote by $G(t,\tau)$ Green's function of system (3.77). It has the form

$$G(t,s) = \begin{cases} \Phi(t,0)(\Phi(s,0))^{-1}, & t \geq s, \\ 0, & t < s. \end{cases} \tag{3.79}$$

Properties of the fundamental matrix yield $G(t,s) = \Phi(t,s)$ for $t \geq s$.

Consider the random process

$$x^*(t) = \int_{-\infty}^{t} \Phi(t,s)\alpha(s)\,ds + \int_{-\infty}^{t} \Phi(t,s)\beta(s)\,dW(s)\,. \tag{3.80}$$

Both integrals in (3.80) exist as seen from the following estimates for $t \in \mathbf{R}$:

$$\int_{-\infty}^{t} ||\Phi(t,s)||\mathbf{E}|\alpha(s)|\,ds \le \int_{-\infty}^{t} K\exp\{-\gamma(t-s)\}\,ds \sup_{t\in\mathbf{R}}(\mathbf{E}|\alpha(t)|^2)^{\frac{1}{2}} < \infty$$

and

$$\int_{-\infty}^{t} ||\Phi(t,s)||^2\mathbf{E}|\beta(s)|^2\,ds \le \int_{-\infty}^{t} K^2\exp\{-2\gamma(t-s)\}\,ds \sup_{t\in\mathbf{R}}\mathbf{E}(|\beta(t)|)^2 < \infty\,.$$

It is clear that $x^*(t)$ is F_t-measurable. Continuity of $x^*(t)$ follows from the evolution property of a matriciant,

$$\Phi(t,s) = \Phi(t,\tau)\Phi(\tau,s),\ t \ge \tau \ge s\,,$$

and the representation of a stochastic integral,

$$\begin{aligned}
&\int_{-\infty}^{t} \Phi(t,s)\beta(s)\,dW(s)\\
&\quad = \int_{-\infty}^{a} \Phi(t,s)\beta(s)\,dW(s) + \int_{a}^{t} \Phi(t,s)\beta(s)\,dW(s)\\
&\quad = \Phi(t,a)\int_{-\infty}^{a} \Phi(a,s)\beta(s)\,dW(s) + X(t)\int_{a}^{t} X^{-1}(s)\beta(s)\,dW(s)\,,
\end{aligned}$$

where $a \in \mathbf{R}$ is a fixed number, and $X(t)$ is a fundamental matrix for system (3.77).

Let us show that $x^*(t)$ satisfies equation (3.75). Indeed, the first integral in (3.80), by estimate (3.78), is uniformly convergent on an arbitrary line segment $[t_1,\,t_2]$. Since

$$\frac{d}{dt}\Phi(t,s) = A(t)\Phi(t,s)$$

and the matrix $A(t)$ is bounded, the formal derivative of the latter integral is also uniformly convergent.

It follows from inequality (3.78) that

$$\int_{-\infty}^{\infty} ||\Phi(t,s)||^2\mathbf{E}|\beta(s)|^2\,ds < \infty$$

and that the integral

$$\int_{-\infty}^{t} ||\Phi(t,s)||^2\mathbf{E}|\beta(s)|^2\,ds$$

converges uniformly in $t \in [t_1, t_2]$. This means that conditions of Lemma 3.2 are satisfied, so the stochastic differential for $x^*(t)$ has the form

$$\begin{aligned} dx^*(t) = &\Big(\int_{-\infty}^t A(t)\Phi(t,s)\alpha(s)\,ds + \Phi(t,t)\alpha(t)\Big)dt \\ &+ \Big(\int_{-\infty}^t A(t)\Phi(t,s)\beta(s)dW(s)\Big)dt + \Phi(t,t)\beta(t)dW(t) \\ &= \Big[A(t)\Big(\int_{-\infty}^t \Phi(t,s)\alpha(s)\,ds + \int_{-\infty}^t \Phi(t,s)\beta(s)dW(s)\Big) + \alpha(t)\Big]dt \\ &+ \beta(t)dW(t) = [A(t)x^*(t) + \alpha(t)]dt + \beta(t)dW(t)\,. \end{aligned}$$

This identity means that (3.80) satisfies system (3.75).

Let us now show that $x^*(t)$ is bounded. To this end, we estimate each term in (3.80). Using the Cauchy–Bunyakovskii inequality we have

$$\begin{aligned} \mathbf{E}\Big|\int_{-\infty}^t \Phi(t,s)\alpha(s)\,ds\Big|^2 &\le \mathbf{E}\Big(\int_{-\infty}^t ||\Phi(t,s)|||\alpha(s)|\,ds\Big)^2 \\ &\le K^2\mathbf{E}\Big(\int_{-\infty}^t \exp\Big\{-\frac{\gamma}{2}(t-s)\Big\}\exp\Big\{-\frac{\gamma}{2}(t-s)\Big\}|\alpha(s)|\,ds\Big)^2 \\ &\le K^2\int_{-\infty}^t \exp\{-\gamma(t-s)\}\,ds\int_{-\infty}^t \exp\{-\gamma(t-s)\}\mathbf{E}|\alpha(s)|^2\,ds \\ &\le K^2\frac{1}{\gamma^2}\sup_{t\in\mathbf{R}}\mathbf{E}|\alpha(t)|^2 < \infty\,. \end{aligned}$$

Properties of a stochastic integral yield

$$\begin{aligned} \mathbf{E}\Big|\int_{-\infty}^t \Phi(t,s)\beta(s)\,dW(s)\Big|^2 &\le \int_{-\infty}^t K^2\exp\{-2\gamma(t-s)\}\sup_{t\in\mathbf{R}}\mathbf{E}|\beta(s)|^2\,ds \\ &\le K^2\frac{1}{2\gamma}\sup_{t\in\mathbf{R}}\mathbf{E}|\beta(t)|^2 < \infty\,. \end{aligned}$$

These relations, together with boundedness of the second moments for $\alpha(t)$ and $\beta(t)$, give the estimate

$$\sup_{t\in\mathbf{R}}\mathbf{E}|x^*(t)|^2 \le C$$

for some constant $C > 0$.

Let us finally show that the solution $x^*(t)$ is mean square exponentially stable. Let $x(t)$ be an arbitrary solution of system (3.75) such that $x(0) = x_0$, where x_0 is a random F_0-measurable variable and $\mathbf{E}|x_0|^2 < \infty$. Then $x(t)$ has the following representation for $t \geq 0$:

$$x(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,s)\alpha(s)\,ds + \int_0^t \Phi(t,s)\beta(s)\,dW(s) \tag{3.81}$$

that can easily be obtained by taking the stochastic differential and using uniqueness of the solution.

We have

$$\mathbf{E}|x(t) - x^*(t)|^2 \leq ||\Phi(t,0)||^2 \mathbf{E}|x^*(0) - x_0|^2 \leq K^2 \exp\{-2\gamma t\}\mathbf{E}|x^*(0) - x_0|^2\,,$$

which implies exponential stability of $x^*(t)$.

Let us prove that it is strongly unique. Let $y(t)$ be a mean square bounded solution of system (3.75) on $\mathbf{R}$, distinct from $x^*(t)$. Then $z(t) = x^*(t) - y(t)$ is a solution of system (3.77). Since it is exponentially stable, $z(t)$ satisfies the estimate

$$|z(t)| \leq K \exp\{\gamma(t-\tau)\}|z(\tau)|$$

with probability 1 for arbitrary $t \geq \tau$. It is clear that $\sup_{t\in\mathbf{R}} \mathbf{E}|z(t)|^2 = a < \infty$ and, hence,

$$\mathbf{E}|z(t)|^2 \leq K^2 \exp\{-2\gamma(t-\tau)\}a$$

for arbitrary $t,\ \tau \in \mathbf{R}, t \geq \tau$. By passing to limit in the last inequality as $\tau \to -\infty$, we see that $\mathbf{E}|z(t)|^2 = 0$ for arbitrary $t \in \mathbf{R}$. This means that

$$\mathbf{P}\{x^*(t) \neq y(t)\} = 0,\ \forall t \in \mathbf{R}\,,$$

and, since the processes $x^*(t)$ and $y(t)$ are continuous, we have trajectory-wise uniqueness, so that

$$\mathbf{P}\{\sup_{t\in\mathbf{R}} |x^*(t) - y(t)| > 0\} = 0\,,$$

which finishes the proof. □

Let us now look at the existence of periodic solutions of system (3.75).

Let the matrix $A(t)$ be periodic in t with period T, and the random processes $\alpha(t)$ and $\beta(t)$ be such that the process

$$\eta(t) = (\alpha(t), \beta(t), W(t+a) - W(t))$$

is periodic in the restricted sense for arbitrary $a \in \mathbf{R}$, that is, finite dimensional distributions of the process are periodic with period T. Let us show that the solution $x^*(t)$ defined by formula (3.80) is T-periodic.

Since the matrix $A(t)$ is T-periodic, the matriciant of system (3.77) satisfies the relation

$$\Phi(t+T, s+T) = \Phi(t, s)\,, \tag{3.82}$$

which follows from the evident relations

$$\begin{aligned}\Phi(t+T, s+T) &= \Phi(t+T, 0)(\Phi(s+T, 0)^{-1} = \Phi(t, 0)\Phi(T, 0) \\ &\times (\Phi(T, 0))^{-1}(\Phi(s, 0))^{-1} = \Phi(t, s)\,.\end{aligned}$$

We will show that each term in (3.80) is a T-periodic random process. Indeed, by (3.78) and (3.82), T-periodicity follows from property 7^0 in [41, p. 184]. To prove that the second term in (3.80) is periodic, it is sufficient to show that the integral

$$\eta_n(t) = \int_{t-n}^{t} \Phi(t, s)\beta(s)\, dW(s)$$

is periodic for every natural $n \geq 1$. Making a change of variables, the integral becomes

$$\eta_n(t) = \int_{-n}^{0} \Phi(t, t+s)\beta(t+s)\, dW(t+s)\,.$$

Now, the proof of periodicity follows that in [41, p. 186] noting that the process $\beta(t+s)$ is periodic in t and the function $\Phi(t, t+s)$ is also periodic in t by (3.82).

This proves periodicity of the two processes that enter in the right-hand side of (3.80) and, since $\alpha(t), \beta(t)$ and $W(t+a) - W(t)$ are periodically connected, the process $x^*(t)$ is also T-periodic.

This proves the following corollary of Theorem 3.6.

Corollary 3.1. *If the matrix $A(t)$ in system (3.75) is T-periodic and the process $\eta(t) = (\alpha(t), \beta(t), W(t+a) - W(t))$ is T-periodic in the restricted sense, and the conditions of Theorem 3.6 are satisfied, then the solution $x^*(t)$ defined by formula (3.80) is a random process that is T-periodic in the restricted sense, and if the matrix $A(t)$ is constant and the process $\eta(t)$ is stationary in the restricted sense, then the solution $x^*(t)$ is stationary.*

3.4 Quasilinear systems

Consider now a more general case where the stochastic system has the form

$$dx = [A(t)x + f(t, x)]dt + \ g(t, x)dW(t)\,, \tag{3.83}$$

where $A(t)$ is a matrix continuous and bounded on $\mathbf{R}$, the functions $f(t,x),\ g(t,x)$ are defined and continuous for $t \in \mathbf{R},\ x \in \mathbf{R}^n$, and Lipschitz continuous in $x \in \mathbf{R}^n$ with a constant L. Let also the functions $f(t,0),\ g(t,0)$ be bounded on $\mathbf{R}$ with some constant N.

Theorem 3.7. *If system* (3.77) *is exponentially stable, then system* (3.83) *has a unique solution that is mean square bounded on the axis, if the Lipschitz constant is sufficiently small. This solution is mean square exponentially stable.*

If the functions A, f, and g are periodic in t with period T, then this solution is T-periodic.

If A, f, and g do not depend on t, then the solution is stationary.

Proof. We will look for a bounded solution of equation (3.83) as the limit of a sequence $\{x_m(t)\}$ by defining $\{x_{m+1}(t)\}$ to be the unique solution, mean square bounded on $\mathbf{R}$, of the linear system

$$dx = [A(t)x + f(t, x_m(t))]dt + \ g(t, x_m(t))dW(t),\ x_0(t) = 0\,, \tag{3.84}$$

which exists by Theorem 3.6, since

$$\begin{aligned}\mathbf{E}|f(t,x_m(t))|^2 &\le 2(L^2\mathbf{E}|x_m(t)|^2 + N^2)\,,\\ \mathbf{E}|g(t,x_m(t))|^2 &\le 2(L^2\mathbf{E}|x_m(t)|^2 + N^2)\,,\end{aligned}$$

and $x_m(t)$ is F_t-measurable, so that the inhomogeneities in the right-hand side of (3.84) are of the class B. By this theorem, a bounded solution has the form

$$x_{m+1}(t) = \int_{-\infty}^{t} \Phi(t,s)f(s,x_m(s))\,ds + \int_{-\infty}^{t} \Phi(t,s)g(s,x_m(s))dW(s)\,. \tag{3.85}$$

Similarly to [41], one can prove convergence, uniform on any bounded line segment $[t_1,\ t_2]$ with probability 1, as well as uniform boundedness of

$$\mathbf{E}|x_m(t)|^2 \le C$$

for arbitrary $t \in \mathbf{R}$, if the Lipschitz constant is sufficiently small. By Fatou's lemma, we obtain the inequality

$$\mathbf{E}|x_\infty(t)|^2 \le C\,.$$

By passing to the limit as $m \to \infty$ in (3.85) and using properties of a stochastic integral and continuity of the functions f and g, we see that the limit random process $x_\infty(t)$ satisfies the integral equation

$$dx_\infty(t) = \int_{-\infty}^{t} \Phi(t,s)f(s,x_\infty(s))\,ds + \int_{-\infty}^{t} \Phi(t,s)g(s,x_\infty(s))dW(s)\,. \tag{3.86}$$

Differentiating it and using Lemma 3.2 we see that the limit process $x_\infty(t)$ satisfies system (3.83). It is clear that it is F_t-measurable, since it is a limit of F_t-measurable processes. Uniqueness of such a process is verified as it is done in [41, p. 272].

Let us show that the obtained solution is mean square stable. Let $y(t)$ be another solution of system (3.83). Then, in the same way as in [41, p. 273], one can prove that it satisfies the relations

$$y(t) = \Phi(t,0)y(0) + \int_0^t \Phi(t,s)f(s,y(s))\,ds + \int_0^t \Phi(t,s)g(s,y(s))dW(s)\,, \quad (3.87)$$

and since $x_\infty(t)$ satisfies a similar relation, we have that

$$\begin{aligned}\mathbf{E}|x_\infty(t) - y(t)|^2 \le 3\Bigg[&K\exp\{-\gamma t\}\mathbf{E}|x_\infty(0) - y(0)|^2 \\ &+ \frac{K^2L^2}{\gamma}\int_0^t \exp\{-\gamma(t-s)\}\mathbf{E}|x_\infty(s) - y(s)|^2\,ds \\ &+ K^2L^2\int_0^t \exp\{-\gamma(t-s)\}\mathbf{E}|x_\infty(s) - y(s)|^2\,ds\Bigg].\end{aligned}$$

We get

$$u(t) \le 3\left[K\mathbf{E}|x_\infty(0) - y(0)|^2 + \left(\frac{K^2L^2}{\gamma} + K^2L^2\right)\int_0^t u(s)\,ds\right],$$

where $u(t) = \exp\{\gamma t\}\mathbf{E}|x_\infty(t) - y(t)|^2$. Using now the Gronwall–Bellman inequality we get

$$u(t) \le 3K\mathbf{E}|x_\infty(0) - y(0)|^2 \exp\left\{\left(\frac{K^2L^2}{\gamma} + K^2L^2\right)t\right\}.$$

Or

$$\mathbf{E}|x_\infty(t) - y(t)|^2 \le 3K\mathbf{E}|x_\infty(0) - y(0)|^2 \exp\left\{\left(\frac{K^2L^2}{\gamma} + K^2L^2 - \gamma\right)t\right\}. \quad (3.88)$$

Take the Lipschitz constant so small that

$$\frac{K^2L^2}{\gamma} + K^2L^2 - \gamma < 0\,.$$

Then inequality (3.88) shows that $x_\infty(t)$ is mean square totally exponentially stable.

A proof of the last part of the theorem about solutions being periodic (stationary) can be obtained from the fact that, by Theorem 3.6, $x_\infty(t)$ is a limit of a sequence of periodic, or stationary, random processes $x_m(t)$ defined by the linear nonhomogeneous system (3.84). Existence of another periodic (stationary) solution of system (3.83) contradicts uniqueness of a mean square bounded solution. $\square$

3.5 Linear system solutions that are probability bounded on the axis. A generalized notion of a solution

Theorem 3.6 asserts that a nonhomogeneous linear system has solutions that are mean square bounded on $\mathbf{R}$ if the corresponding homogeneous system is exponentially stable. As opposed to the deterministic system, one can not infer existence of a bounded solution in the dichotomous case. The reason for this lies in the definition of a solution of the stochastic Ito equation, i.e., in the requirement that it needs to agree with the corresponding flow of σ-algebras, making formula (3.27), which is an analogue of formula (3.24) that gives a representation of a bounded solution in terms of Green's formula, to loose sense. However, by broadening the notion of a solution and dropping the requirement that it should be F_t-measurable, one can obtain a dichotomy result similar to the deterministic case. Corresponding results have already been obtained in the cited work [61]. To make the exposition of the problem complete, we give these results.

Consider the a system of stochastic differential equations

$$dx(t) = (Ax(t) + f(t))\,dt + \sum_{k=1}^{m}(B_k x(t) + g_k(t))\,dw_k(t)\,, \tag{3.89}$$

where A and B_k are real $n \times n$-matrices, $f(t) = \big(f_1(t), f_2(t), \dots, f_n(t)\big)$ and $g_k(t) = \big(g_{k1}(t), g_{k2}(t), \dots, g_{kn}(t)\big)$ are vector-valued functions, $x(t) = \big(x_1(t), x_2(t), \dots, x_n(t)\big)$ is a solution, $t \in \mathbf{R}$, $w_k(t)$ are independent scalar Wiener processes, $t \in \mathbf{R}$, and $k = \overline{1, m}$. A Wiener process $w(t)$, $t \in \mathbf{R}$, is understood as a process with independent increments and such that $w(t) - w(s)$ is a Gaussian random variable for arbitrary $s, t \in \mathbf{R}$,

$$w(0) = 0, \quad \mathbf{E}\big(w(t) - w(s)\big) = 0, \quad \mathbf{E}\big(w(t) - w(s)\big)^2 = |\,t - s|\,.$$

Introduce the following family of σ-algebras for $-\infty \le a \le b \le +\infty$:

$$\mathcal{F}_a^b = \sigma\big\{w_k(s_2) - w_k(s_1) : a \le s_1 \le s_2 \le b, k = \overline{1,m}\big\}\,.$$

Let $x_s(z;t)$, $z \in \mathbf{R}^n$, $s \le t$, be a process such that

$$x_s(z;t) = z + \int_s^t \big(Ax_s(z;u) + f(u)\big)\,du + \sum_{k=1}^m \int_s^t \big(B_k x_s(z;u) + g_k(u)\big)\,dw_k(u)\,.$$

It is well known [89] that a solution of the Cauchy problem for (3.89) defines a continuous stochastic flow and there is a modification of it such that $x_s(\,\cdot\,;t)$ is a homeomorphism into $\mathbf{R}^n$ for almost all $\omega \in \Omega$. By a solution of (3.89), we understand a random process that agrees with this flow. A rigorous definition is the following.

Definition 3.2. A process $x(t)$, $t \in \mathbb{R}$, is called a *solution* of system (3.89) if

$$x_s\big(x(s);t\big) = x(t)$$

$\mathbf{P}$-almost certainly for all $-\infty < s \le t < +\infty$.

Definition 3.2 shows that $x_s^{-1}\big(x(t);t\big) = x(s)$ $\mathbf{P}$-almost certainly for all $-\infty < s \le t < +\infty$.

Definition 3.3. A solution $x(t)$, $t \in \mathbf{R}$, of system (3.89) is called *stochastically bounded* if

$$\lim_{N\to+\infty} \sup_{t\in\mathbb{R}} \mathbf{P}\{|\,x(t)| > N\} = 0\,.$$

The process $x_s(z;t)$, $s \le t$, according to [186], can be represented as

$$\begin{aligned} x_s(z;t) = H_s^t\Bigg[z &+ \int_s^t \big(H_s^u\big)^{-1}\Big(f(u) - \sum_{k=1}^m B_k g_k(u)\Big)\,du \\ &+ \sum_{k=1}^m \int_s^t \big(H_s^u\big)^{-1} g_k(u)\,dw_k(u)\Bigg], \end{aligned} \tag{3.90}$$

where H_s^t is a stochastic semigroup, $H_s^t H_r^s = H_r^t$, $r \le s \le t$, satisfying the homogeneous matrix equation [156],

$$dH_s^t = AH_s^t dt + \sum_{k=1}^m B_k H_s^t dw_k(t), \quad H_s^s = I, \quad s \le t\,. \tag{3.91}$$

The asymptotic behavior of H_s^t will be studied as in [156, p. 226]. Set

$$L_1 = \big\{x : \lim_{t\to+\infty} H_s^t x = 0 \ (\mathbf{P} = 1)\big\},$$
$$L_2 = \big\{x : \lim_{s\to-\infty} \big(H_s^t\big)^{-1} x = 0 \ (\mathbf{P} = 1)\big\}.$$

Lemma 3.3. *The sets L_1 and L_2 are linear invariant subspaces with respect to the semigroup H_s^t.*

Proof. Consider $L_1 = L_1(s)$. For each $s \in \mathbf{R}$, set

$$\xi_m(s) = \sup_{m\le t\le (m+1)} |H_s^{s+t} x|, \quad m \in \mathbb{Z}^+ .$$

Since $\xi_m(s)$ and $\xi_m(0)$ have the same distributions, we have that

$$\mathbf{P}\left\{\lim_{t\to+\infty} H_s^t x = 0\right\} = \mathbf{P}\left\{\bigcap_{k=1}^{\infty}\bigcup_{N=1}^{\infty}\left(\sup_{N\le t}\left|H_s^{s+t}x\right| \le k^{-1}\right)\right\}$$
$$= \mathbf{P}\left\{\bigcap_{k=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{m=N}^{\infty}\left(\xi_m(s) \le k^{-1}\right)\right\} = \mathbf{P}\left\{\bigcap_{k=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{m=N}^{\infty}\left(\xi_m(0) \le k^{-1}\right)\right\}$$
$$= \mathbf{P}\left\{\bigcap_{k=1}^{\infty}\bigcup_{N=1}^{\infty}\left(\sup_{N\le t}\left|H_0^t x\right| \le k^{-1}\right)\right\} = \mathbf{P}\left\{\lim_{t\to+\infty} H_0^t x = 0\right\},$$

hence, $L_1(s) = L_1(0) = L_1$ does not depend on $s \in \mathbf{R}$. For $x \in L_1$ and $u \ge s$, we have

$$1 = \mathbf{P}\Big\{\lim_{t\to+\infty} H_s^t x = 0\Big\} = \mathbf{P}\Big\{\lim_{t\to+\infty} H_s^{u+t} x = 0\Big\}$$
$$= \mathbf{P}\Big\{\lim_{t\to+\infty} H_s^u H_u^{u+t} x = 0\Big\} = \int \mathbf{P}\big\{H_s^u x \in dz\big\}\mathbf{P}\Big\{\lim_{t\to+\infty} H_u^{u+t} z = 0\Big\}.$$

Thus, $\mathbf{P}\big\{\lim_{t\to+\infty} H_u^t z = 0\big\} = 1$. So, $z = H_s^u x$ almost everywhere with respect to the measure $\mathbf{P}\big\{H_s^u x \in dz\big\}$. This implies invariance of L_1 with respect to H_s^t. Linearity of L_1 follows from linearity of H_s^t. For the subspace L_2, the proof is the same as for L_1. ☐

Let P_i be projections on the subspaces L_i, $i = 1, 2$. It follows from [70, 156] that the semigroup H_s^t is exponentially p-stable on the subspace L_1, and the semigroup $\big(H_s^t\big)^{-1}$ is exponentially p-unstable on the subspace L_2, so that there exist constants $p_1 > 0$ and $p_2 > 0$ such that, for some $D_i = D_i(p) > 0$, $\lambda_i = \lambda_i(p) > 0$, we have

$$\sup_{|x|=1} \mathbf{E}\big|H_s^t P_1 x\big|^p \le D_1 e^{-\lambda_1(t-s)}, \ p \in (0, p_1)\,, \tag{3.92}$$

$$\sup_{|x|=1} \mathbf{E}\big|\big(H_s^t\big)^{-1} P_2 x\big|^p \le D_2 e^{-\lambda_2(t-s)}, \ p \in (0, p_2)\,. \tag{3.93}$$

Set

$$p_0 = \min\{p_1, p_2\}.$$

For $p \in (0, p_0)$, the semigroups H_s^t and $\big(H_s^t\big)^{-1}$ are exponentially p-stable on the subspaces L_1 and L_2, correspondingly.

Remark. The problem of finding the quantity $\hat{p}$ such that the semigroup H_s^t is exponentially p-stable for $p \in (0, \hat{p})$ and H_s^t is p-exponentially unstable for $p \in (\hat{p}, +\infty)$ was studied in [10]. It was shown there that $\hat{p}$ satisfies the equation

$$g(p\,) = 0, \ \text{where } g(p\,) := \lim_{t\to+\infty} t^{-1} \ln \mathbf{E}\big|\, H_s^t x\big|^p\,.$$

Lemma 3.4. *The semigroup H_s^t is exponentially p-unstable on the subspace L_2, so that*

$$\sup_{|x|=1} \mathbf{E}\big|H_s^t P_2 x\big|^{-p} \le D_2 e^{-\lambda_2(t-s)}, \ p \in (0, p_2)\,.$$

Proof. For $y_s(t) = H_s^t P_2 x / |H_s^t P_2 x|$, we have

$$\big|H_s^t P_2 x\big|^{-1} = \big|\big(H_s^t\big)^{-1} y_s(t)\big| |P_2 x|^{-1}\,. \tag{3.94}$$

It follows from (3.93) and (3.94) that

$$\mathbf{E}\big|H_s^t P_2 x\big|^{-p} \le \sup_{|y|=1} \mathbf{E}\big|\big(H_s^t\big)^{-1} P_2 y\big|^p\, |P_2 x|^{-p} \le D_2 e^{-\lambda_2(t-s)} |P_2 x|^{-p}\,.$$

□

Since system (3.89) is linear and the subspace L_2 is invariant with respect to the semigroup H_s^t, from the preceding lemma and [70, p. 237] it follows that $\lim_{t\to+\infty} |H_s^t x| = +\infty (\mathbf{P} = 1)$ for $x \in L_2$. This means that $L_1 \cap L_2 = \{0\}$.

The following auxiliary lemmas will be needed to prove the main result in the section.

Lemma 3.5. *Let $\varphi(t)$ be a continuous function, $t \in \mathbf{R}$. Then the following change of the direction of integration in a stochastic integral holds for $s \le t$:*

$$H_s^t \int_s^t \big(H_s^u\big)^{-1} \varphi(u)\, dw_k(u) = -\int_t^s H_u^t \varphi(u)\, dw_k(u) - \int_t^s H_u^t B_k \varphi(u)\, du\,. \tag{3.95}$$

Proof. The proof is conducted in a standard way using the definition of a stochastic integral, going in the direct and the reverse directions. □

Lemma 3.6. *Let $\varphi(t)$ be a continuous bounded function such that $\sup_{t\in\mathbf{R}}|\varphi(t)| < K < +\infty$. Then for $r = 1, 2$ and $0 < p < p_0$ there exist $T > 0$ and $0 < q < 1$ such that the following inequalities hold for arbitrary $t \in \mathbf{R}$, $N > 0$, and $n \in \mathbf{Z}^+$:*

$$\mathbf{P}\Big\{\int_{t-T(n+1)}^{t-Tn} \big|\, H_u^t P_1\varphi(u)\big|^r du > N^r 2^{-n}\Big\} \le L_1 N^{-p} q^n\,,$$

$$\mathbf{P}\Big\{\int_{t+Tn}^{t+T(n+1)} \big|\, \big(H_t^u\big)^{-1} P_2\varphi(u)\big|^r du > N^r 2^{-n}\Big\} \le L_1 N^{-p} q^n\,.$$

Proof. Using (3.92) we have a chain of inequalities for $0 < p < p_0$,

$$\begin{aligned}
&\mathbf{P}\Big\{\int_{t-T(n+1)}^{t-Tn} \big|\, H_u^t P_1\varphi(u)\big|^r du > N^r 2^{-n}\Big\}\\
&\quad\le \mathbf{P}\Big\{\int_0^T \big|\, H_{t-Tn-u}^t P_1\varphi(t-Tn-u)\big|^r du > N^r 2^{-n}\Big\}\\
&\quad\le \mathbf{P}\Big\{T \sup_{0\le u\le T} \big|\, H_0^{Tn+u} P_1\varphi(t-Tn-u)\big|^r > N^r 2^{-n}\Big\}\\
&\quad\le \big(N^{-r}2^n T K^r\big)^{p/r} \sup_{|x|\le 1} \mathbf{E}\Big(\sup_{0\le u\le T} \big|\, H_{Tn}^{Tn+u} H_0^{Tn} P_1 x\big|\Big)^p\\
&\quad\le \big(N^{-r}2^n T K^r\big)^{p/r} \sup_{|y|\le 1} \mathbf{E}\Big(\sup_{0\le u\le T} \big|\, H_0^u P_1 y\big|\Big)^p \sup_{|x|\le 1} \mathbf{E}\big(\big|\, H_0^{Tn} P_1 x\big|\big)^p\\
&\quad\le L\big(D_1 T K^r\big)^p N^{-p} \exp\big\{\big(-\lambda_1 + T^{-1}\ln 2\big)Tnp\big\}\,,
\end{aligned}$$

where $L = \sup_{|y|=1} \mathbf{E}\big(\sup_{0\le u\le T} \big|\, H_0^u P_1 y\big|\big)^p < +\infty$. It remains to choose a sufficiently large $T > 0$ and set $L_1 = L(D_1 T K^r)^p$, $q = \exp\big\{\big(-\lambda_1 + T^{-1}\ln 2\big)Tnp\big\}$.

The second inequality is proved similarly. □

Lemma 3.7. *Let $\varphi(t)$ be a continuous bounded function such that $\sup_{t\in\mathbf{R}}|\varphi(t)| < K < +\infty$. Then the following limits exist with probability 1 for arbitrary $t \in \mathbf{R}$:*

$$\lim_{s\to-\infty} \int_s^t H_u^t P_1\varphi(u)\,du = \int_{-\infty}^t H_u^t P_1\varphi(u)\,du,$$

$$\lim_{s\to+\infty}\int_t^s \left(H_t^u\right)^{-1} P_2\varphi(u)\,du = \int_t^{+\infty} \left(H_t^u\right)^{-1} P_2\varphi(u)\,du\,.$$

Proof. For proving the first identity, let us show that the series

$$\sum_{n=1}^{\infty}\int_{t-T(n+1)}^{t-Tn} |\,H_u^t P_1\varphi(u)|\,du\,,$$

$T>0$, converges with probability 1. By the Borel–Cantelli lemma it is sufficient that

$$\sum_{n=1}^{\infty}\mathbf{P}\left\{\int_{t-T(n+1)}^{t-Tn}\left|\,H_u^t P_1\varphi(u)\right|du > 2^{-n}\right\} < +\infty\,. \tag{3.96}$$

The latter inequality follows from Lemma 3.6.

The second identity is proved similarly. □

Lemma 3.8. *Let $\varphi(t)$ be a continuous bounded function such that $\sup_{t\in\mathbf{R}}|\,\varphi(t)| < K < +\infty$. Then the following limits exist with probability* 1 *for arbitrary $t\in\mathbf{R}$:*

$$\lim_{s\to-\infty}\int_t^s H_u^t P_1\varphi(u)\,dw_k(u) = \int_t^{-\infty} H_u^t P_1\varphi(u)\,dw_k(u),$$

$$\lim_{s\to+\infty}\int_t^s \left(H_t^u\right)^{-1} P_2\varphi(u)\,dw_k(u) = \int_t^{+\infty} \left(H_t^u\right)^{-1} P_2\varphi(u)\,dw_k(u)\,.$$

Proof. Let

$$m_{ki}^t(v) = \int_t^{t-v}\left(H_u^t P_1\varphi(u)\right)_i dw_k(u), \quad i=\overline{1,n},$$

be components of the stochastic vector integral $\int_t^{t-v} H_u^t P_1\varphi(u)\,dw_k(u)$, $v\ge 0$. These components are martingales with the characteristics

$$\langle m_{ki}^t\rangle(v) = -\int_t^{t-v}\left(H_u^t P_1\varphi(u)\right)_i^2 du = \int_0^v\left(H_{t-u}^t P_1\varphi(t-u)\right)_i^2 du, \quad v\ge 0$$

with respect to the family of the σ-algebras $\mathcal{F}_{t-v}^t$, $v\ge 0$. As in the preceding lemma, it sufficient to verify existence of the limits

$$\langle m_{ki}^t\rangle(+\infty) = \lim_{v\to+\infty}\langle m_{ki}^t\rangle(v) = \int_0^{+\infty}\left(H_{t-u}^t P_1\varphi(t-u)\right)_i^2 du.$$

However, the square integrable martingale $m_{ki}^t(v)$ is closed, which proves the first identity of the lemma. The second one is proved similarly. □

Definition 3.4. The operator-valued function G_s^t,

$$G_s^t = \begin{cases} H_s^t P_1, & \text{if } s < t, \\ -\left(H_t^s\right)^{-1} P_2, & \text{if } t < s, \end{cases} \tag{3.97}$$

will be called *Green's function* for system (3.89).

Now we give the main result in this section.

Theorem 3.8. *Let $L_1 \oplus L_2 = \mathbf{R}^n$. Then, for arbitrary continuous bounded functions $f(t)$, $g_k(t)$, $k = \overline{1, m}$, $t \in \mathbf{R}$, system (3.89) has a unique solution $\tilde{x}(t)$ that is stochastically bounded on $\mathbf{R}$. This solution has the form*

$$\tilde{x}(t) = \int_{-\infty}^{+\infty} G_u^t \left(f(u) - \sum_{k=1}^{m} B_k P_2 g_k(u) \right) du$$
$$+ \sum_{k=1}^{m} \left(- \int_t^{-\infty} G_u^t g_k(u)\, dw_k(u) + \int_t^{+\infty} G_u^t g_k(u)\, dw_k(u) \right). \tag{3.98}$$

For $0 < p < p_0$, we have the following estimate:

$$\sup_{t \in \mathbf{R}} \mathbf{E} |\, \tilde{x}(t)|^p < +\infty\,. \tag{3.99}$$

Proof. Since $H_s^t = H_s^t P_1 + H_s^t P_2$ and system (3.89) is linear, to prove the theorem it is sufficient to consider the subspaces L_1 and L_2 separately.

With a use of (3.95), identity (3.90) can be written on the subspace L_1 in the form

$$P_1 x_s(z;t) = H_s^t P_1 z - \int_t^s H_u^t P_1 f(u)\, du - \sum_{k=1}^{m} \int_t^s H_u^t P_1 g_k(u)\, dw_k(u), \;\; s \le t\,. \tag{3.100}$$

Setting $P_1 z = 0$ in (3.100) and passing to limit as $s \to -\infty$ we get

$$P_1 \tilde{x}(t) = - \int_t^{-\infty} H_u^t P_1 f(u)\, du - \sum_{k=1}^{m} \int_t^{-\infty} H_u^t P_1 g_k(u)\, dw_k(u)\,. \tag{3.101}$$

It follows from Lemmas 3.7 and 3.8 that the right-hand side of identity (3.101) exists for arbitrary $t \in \mathbf{R}$. The process $P_1 \tilde{x}(t)$ satisfies (3.100) and is a solution of system (3.89) on the subspace L_1. It is clear that $P_1 \tilde{x}(t)$ is $\mathcal{F}_t$-measurable.

Let us now show that it is stochastically bounded. In view of the Chebyshev inequality, it is sufficient to prove inequality (3.99). It follows from (3.101) that

$$\mathbf{P}\Big\{|P_1\tilde{x}(t)| > N\Big\} \leq \mathbf{P}\Big\{\Big|\int_t^{-\infty} H_u^t P_1 f(t)du\Big| > N/m+1\Big\}$$
$$+ \sum_{k=1}^{m} \mathbf{P}\Big\{\Big|\int_t^{-\infty} H_u^t P_1 g_k(u)dw_k(u)\Big| > N/m+1\Big\}. \quad (3.102)$$

The first integral in (3.102) satisfies the estimate

$$\mathbf{P}\Big\{-\int_t^{-\infty} \big|\, H_u^t P_1 f(u)\big|^r du > N^r\Big\} \leq L_2 N^{-p}\,, \quad (3.103)$$

where $r = 1, 2$, $0 < p < p_0$, and $L_2 = L_2(p) < +\infty$. Indeed, using Lemma 3.6 we obtain

$$\mathbf{P}\Big\{-\int_t^{-\infty} \big|\, H_u^t P_1 f(u)\big|^r du > N^r\Big\}$$
$$\leq \mathbf{P}\Big\{\bigcup_{n=0}^{\infty}\Big(\int_{t-T(n+1)}^{t-Tn} \big|\, H_u^t P_1 f(u)\big|^r du > N^r 2^{-(n+1)}\Big)\Big\}$$
$$\leq \sum_{n=0}^{\infty} \mathbf{P}\Big\{\int_{t-T(n+1)}^{t-Tn} \big|\, H_u^t P_1 f(u)\big|^r du > N^r 2^{-(n+1)}\Big\}$$
$$\leq 2^p L_1 (1-q)^{-1} N^{-p}\,.$$

The stochastic integrals in (3.102) satisfy the following estimate:

$$\mathbf{P}\Big\{\Big|\int_t^{-\infty} H_u^t P_1 g_k(u)\, dw_k(u)\Big| > N\Big\} \leq L_3 N^{-p}\,, \quad (3.104)$$

where $0 < p < p_0$ and $L_3 = L_3(p) < +\infty$. To prove (3.104), we will use (3.8), where we set $\varphi(u) = g_k(u)$. Then, by [195], we get

$$\mathbf{P}\Big\{\Big|\int_t^{-\infty} H_u^t P_1 g_k(u)\, dw_k(u)\Big| > N\Big\}$$
$$\leq \sum_{i=1}^{n} \mathbf{P}\Big\{\Big|\int_t^{-\infty} \big(H_u^t P_1 g_k(u)\big)_i\, dw_k(u)\Big| > (N^2 n^{-1})^{1/2}\Big\}$$
$$\leq \sum_{i=1}^{n} \lim_{V\to+\infty} \mathbf{P}\Big\{\sup_{0<v<V} |\, m_{ki}^t(v)| > (N^2 n^{-1})^{1/2}\Big\}$$
$$\leq \sum_{i=1}^{n} \lim_{V\to+\infty} (N^2 n^{-1})^{-p/2}\, \mathbf{E}\Big(\sup_{0<v<V} |\, m_{ki}^t(v)|\Big)^p$$

$$
\leq N^{-p} \sum_{i=1}^{n} n^{p/2} \, c_p \lim_{V \to +\infty} \mathbf{E} \langle m_{ki}^t \rangle^{p/2} (V)
$$

$$
\leq N^{-p} n^{(2+p)/2} \, c_p \, \mathbf{E} \left(- \int_t^{-\infty} \left| H_u^t P_1 g_k(u) \right|^2 du \right)^{p/2}, \quad c_p < +\infty \, .
$$

It remains to prove that $\mathbf{E}\big(-\int_t^{-\infty} \left|H_u^t P_1 g_k(u)\right|^2 du\big)^{p/2} < +\infty$, $t \in \mathbf{R}$, $0 < p < p_0$. By using inequality (3.103) for $p := p + \delta$, $\delta = (p_0 - p)/2$, $f(u) = g_k(u)$, and $r = 2$, we get

$$
\begin{aligned}
&\mathbf{E} \left(- \int_t^{-\infty} \left| H_u^t P_1 g_k(u) \right|^2 du \right)^{p/2} \\
&\quad \leq \sum_{n=0}^{\infty} 2^{p(n+1)} \mathbf{P} \left\{ 4^n < - \int_t^{-\infty} \left| H_u^t P_1 g_k(u) \right|^2 du \leq 4^{n+1} \right\} \\
&\quad \leq \sum_{n=0}^{\infty} 2^{p(n+1)} \mathbf{P} \left\{ 4^n < - \int_t^{-\infty} \left| H_u^t P_1 g_k(u) \right|^2 du \right\} \\
&\quad \leq \sum_{n=0}^{\infty} 2^{p(n+1)} L_2 \, 2^{-(p+\delta)n} \leq \sum_{n=0}^{\infty} L_2 \, 2^p \, 2^{-\delta n} < +\infty \, .
\end{aligned}
$$

Let us finally show that inequality (3.99) holds for $P_1 \tilde{x}(t)$. It follows from (3.102), (3.103), where $r = 1$, and (3.104) that

$$
\mathbf{P} \big\{ | \, P_1 \tilde{x}(t) | > N \big\} \leq L_3 \, N^{-p} \, , \tag{3.105}
$$

where $0 < p < p_0$, $L_3 = L_3(p) < +\infty$. Use (3.105) with $p := p + \delta$, $\delta = (p_0 - p)/2$. Then

$$
\begin{aligned}
\mathbf{E} | \, P_1 \tilde{x}(t) |^p &\leq \sum_{n=0}^{\infty} 2^{p(n+1)} \mathbf{P} \big\{ 2^n < | \, P_1 \tilde{x}(t) | \leq 2^{n+1} \big\} \\
&\leq \sum_{n=0}^{\infty} 2^{p(n+1)} \mathbf{P} \big\{ 2^n < | \, P_1 \tilde{x}(t) | \big\} \\
&\leq \sum_{n=0}^{\infty} 2^{p(n+1)} L_3 \, 2^{-(p+\delta)n} \leq \sum_{n=0}^{\infty} L_3 \, 2^p \, 2^{-\delta n} < +\infty \, .
\end{aligned}
$$

Hence, process (3.101) defines on the subspace L_1 a stochastically bounded solution of equation (3.89), and inequality (3.99) holds.

It remains to show that the solution is unique. Let $y(t)$ be the difference between two such solutions of system (3.89). It is easy to see that $y(t)$ is a stochastically bounded process. On the other hand, $y(t)$ is a solution of the homogeneous system (3.91) on L_1. But

$$\lim_{s\to-\infty} \left|\left(H_s^t\right)^{-1} P_1 z\right| = +\infty \ (\mathbf{P}=1)\,, \tag{3.106}$$

hence, $y(t)$ is trivial.

Thus $P_1\tilde{x}(t)$ is a unique stochastically bounded solution of equation (3.89) on the subspace L_1.

To construct a stochastically bounded solution of equation (3.89) on the subspace L_2, we first solve (3.90) for z, and then project the solution to L_2 and interchange s with t. Denote $x_t^{-1}(y;s) = z$ and $y = x_t(z;s)$. We get

$$\begin{aligned} P_2 x_t^{-1}(y;s) = \left(H_t^s\right)^{-1} P_2 y - \int_t^s \left(H_t^u\right)^{-1} P_2\left(f(u) - \sum_{k=1}^m B_k g_k(u)\right) du \\ - \sum_{k=1}^m \int_t^s \left(H_t^u\right)^{-1} P_2 g_k(u)\, dw_k(u)\,, \quad t \le s\,. \end{aligned} \tag{3.107}$$

The expressions in (3.107) define $P_2 x_t^{-1}(y;s)$ on L_2 in terms of $P_2 y$ and time s. Set $P_2 y = 0$ in (3.107) and pass to the limit for $s \to +\infty$. We get

$$\begin{aligned} P_2\tilde{x}(t) = - \int_t^{+\infty} \left(H_t^u\right)^{-1} P_2\left(f(u) - \sum_{k=1}^m B_k g_k(u)\right) du \\ - \sum_{k=1}^m \int_t^{+\infty} \left(H_t^u\right)^{-1} P_2 g_k(u)\, dw_k(u), \quad t \in \mathbf{R}\,. \end{aligned} \tag{3.108}$$

It follows from Lemmas 3.7 and 3.8 that the expression in the right-hand side of (3.108) exists for $t \in \mathbf{R}$. The process $P_2\tilde{x}(t)$ satisfies (3.107) and, hence, is a solution of system (3.89) on the subspace L_2. Inequality (3.99) for $P_2\tilde{x}(t)$ and its uniqueness are proved as for $P_1\tilde{x}(t)$. The process $P_2\tilde{x}(t)$ is $\mathcal{F}^t$-measurable.

Using (3.97), formula (3.98) follows then from the identity $\tilde{x}(t) = P_1\tilde{x}(t) + P_2\tilde{x}(t)$. The random process $\tilde{x}(t)$ is $\mathcal{F}_{-\infty}^{+\infty}$-measurable. To prove estimate (3.99), it is sufficient to use the inequality

$$\mathbf{P}\left\{|\,\tilde{x}(t)| > N\right\} \le \mathbf{P}\left\{|\,P_1\tilde{x}(t)| > 2^{-1}N\right\} + \mathbf{P}\left\{|\,P_2\tilde{x}(t)| > 2^{-1}N\right\}.$$

□

3.6 Asymptotic equivalence of linear systems

The notion of *asymptotic equivalence* of two systems is well known in the theory of ordinary differential equations. This means that one can establish a one-to-one correspondence between solutions of the two systems such that the difference between corresponding solutions tends to zero for $t \to \infty$. A simple condition for linear systems to be equivalent is given by a theorem of Levinson, see e.g. [38, p. 159].

Theorem 3.9. *Let solutions of system*

$$\frac{dx}{dt} = Ax\,, \tag{3.109}$$

where A is a constant matrix, be bounded on $[0, \infty)$. Then the system

$$\frac{dy}{dt} = [A + B(t)]y\,, \tag{3.110}$$

where $B(t) \in C[0, \infty)$ and

$$\int_0^{\infty} \|B(t)\| dt < \infty,$$

is asymptotically equivalent to system (3.109)*.*

In this section, we study a similar kind of problems for a stochastic system. Namely, we construct a system of ordinary differential equations that is asymptotically equivalent to the system of stochastic equations.

Stochastic systems that have solutions exhibiting similar behavior will be called *similar*, as in the case of ordinary differential equations. It is natural that in a study of stochastic systems there appear notions of asymptotic equivalence in various probability senses. We are aware of only a few works in this directions, see e.g. [87, 30, 31].

Sufficient conditions for asymptotic equivalence of stochastic differential Ito systems and systems of ordinary differential equations in the mean square sense with probability 1 were obtained in [83, 84]. The results obtained in these works make the content of the next two sections.

Consider a system of ordinary differential equations,

$$dx = f(t, x)dt\,, \tag{3.111}$$

where $t \geq t_0 \geq 0, x \in \mathbf{R^n}$, $f(t, x) \in C(\mathbf{R}_+, \mathbf{R^n})$ is an n-dimensional function. Together with system (3.111), consider a system of stochastic differential equations,

$$dy = g(t, y)dt + \sigma(t, y)dW(t)\,, \tag{3.112}$$

where $g(t,x), \sigma(t,y)$ are functions continuous in the totality of the variables, $W(t)$ is a standard scalar Wiener process defined for $t \geq 0$ on a probability space $(\Omega, \mathbf{F}, P)$, $\{\mathcal{F}_t, t \geq 0\}$ is a flow of σ-algebras with which the process $W(t)$ agrees. Then, as it is well known (see, e.g., [186, p. 230]) there are some conditions on the functions $g(t,y), \sigma(t,y)$ such that the system of stochastic differential equations (3.112) has a unique solution $y(t) \equiv y(t,\omega) \in \mathbf{R^n}$ with the initial conditions $y(t_0) = y_0$, $E|y_0|^2 < \infty$.

Definition 3.5. If for each solution $y(t)$ of system (3.112) there corresponds a solution $x(t)$ of system (3.111) such that

$$\lim_{t\to\infty} \mathbf{E}|x(t) - y(t)|^2 = 0\,,$$

then system (3.112) is called *asymptotically mean square equivalent* to system (3.111).

Definition 3.6. If for every solution $y(t)$ of system (3.112) there corresponds a solution $x(t)$ of system (3.111) such that

$$\mathbf{P}\{\lim_{t\to\infty} |x(t) - y(t)| = 0\} = 1\,,$$

then system (3.112) is *asymptotically equivalent* to system (3.111) with probability 1.

Let system (3.111) be of the form

$$dx = Axdt\,, \tag{3.113}$$

with the initial condition $x(t_0) = x_0$, $t \geq t_0 \geq 0, x \in \mathbf{R^n}$, and A being a deterministic constant matrix.

Together with system (3.113), consider a system of stochastic differential equations of the form

$$dy = (A + B(t))ydt + D(t)ydW(t)\,, \tag{3.114}$$

where $B(t), D(t)$ are continuous deterministic matrices.

The following theorem is a generalization of the theorem of Levinson to the case of stochastic differential equations.

Theorem 3.10. *Let all solutions of system* (3.113) *be bounded on* $[0,\infty)$*. If*

$$\begin{gathered}\int_0^\infty \|B(t)\|dt \leq K_1 < \infty\,,\\ \int_0^\infty \|D(t)\|^2 dt \leq K_1\end{gathered} \tag{3.115}$$

for some $K_1 > 0$*, then system* (3.114) *is asymptotically mean square equivalent to system* (3.113)*.*

If the first condition in (3.115) *and the condition*

$$\int_0^\infty t\|D(t)\|^2 dt \le K_1 < \infty \tag{3.116}$$

are satisfied, then system (3.114) *is asymptotically equivalent with probability* 1 *to system* (3.113)*.*

Proof. We split the proof of the first part of the theorem into several steps.

I) Since solutions of system (3.113) are bounded, the eigen values $\lambda(A)$ of the matrix A satisfy the inequality $\operatorname{Re}\lambda(A) \le 0$, and the values that have zero real part have simple elementary divisors.

Without loss of generality, we can assume that the matrix A has a quasidiagonal form,

$$A = \operatorname{diag}(A_1, A_2)\,,$$

where A_1 and A_2 are $(p \times p)$- and $(q \times q)$-matrices correspondingly, $p + q = n$, such that

$$\operatorname{Re}\lambda(A_1) \le -\alpha < 0, \qquad \operatorname{Re}\lambda(A_2) = 0\,. \tag{3.117}$$

This can always be achieved by using a nonsingular transformation

$$\xi = Sx,$$

where S is a nondegenerate $(n \times n)$-matrix.

Let

$$X(t) = \operatorname{diag}(e^{tA_1}, e^{tA_2})$$

be a fundamental matrix of system (3.113), normalized in zero, $X(0) = E$, and

$$I_1 = \operatorname{diag}(E_p, 0), \qquad I_2 = \operatorname{diag}(0, E_q)\,,$$

where E_p and E_q are the identity matrices of orders p and q, correspondingly. It is clear that $I_1 + I_2 = E_n$. Set

$$X(t) = X_1(t) + X_2(t)\,,$$

where

$$X_1(t) = X(t)I_1 = \operatorname{diag}(e^{tA_1}, 0)\,,$$

and

$$X_2(t) = X(t)I_2 = \operatorname{diag}(0, e^{tA_2})\,.$$

Hence, the Cauchy matrix

$$\widetilde{X}(t,\tau) = X(t)X^{-1}(\tau) = X(t-\tau)$$

can be written as

$$\widetilde{X}(t,\tau) = X_1(t-\tau) + X_2(t-\tau)\,. \tag{3.118}$$

Using estimates (3.117) we get

$$\|X_1(t)\| = \|e^{tA_1}\| \le ae^{-\alpha t}, \qquad t \ge t_0 \ge 0\,, \tag{3.119}$$

and

$$\|X_2(t)\| = \|e^{tA_2}\| \le b, \qquad t \in \mathbf{R}\,, \tag{3.120}$$

where a, b, α are some positive constants.

Let us write a solution of system (3.114) with the initial conditions $y(t_0) = y_0$ in terms of a Cauchy matrix for the deterministic differential system (3.113) using [186, p. 234] as

$$\begin{aligned} y(t) = X(t-t_0)y(t_0) &+ \int_{t_0}^{t} X_1(t-\tau)B(\tau)y(\tau)d\tau + \int_{t_0}^{t} X_2(t-\tau)B(\tau)y(\tau)d\tau \\ &+ \int_{t_0}^{t} X_1(t-\tau)D(\tau)y(\tau)dW(\tau) + \int_{t_0}^{t} X_2(t-\tau)D(\tau)y(\tau)dW(\tau) \end{aligned} \tag{3.121}$$

for $t \ge t_0 \ge 0$.

Using the evolution properties of the matriciant

$$X_2(t-\tau) = X(t-\tau)I_2 = X(t-t_0)X(t_0-\tau)I_2 = X(t-t_0)X_2(t_0-\tau)\,,$$

rewrite (3.121) as follows:

$$\begin{aligned} y(t) = X(t-t_0)\Bigg[y(t_0) &+ \int_{t_0}^{\infty} X_2(t_0-\tau)B(\tau)y(\tau)d\tau \\ &+ \int_{t_0}^{\infty} X_2(t_0-\tau)D(\tau)y(\tau)dW(\tau)\Bigg] \\ &+ \int_{t_0}^{t} X_1(t-\tau)B(\tau)y(\tau)d\tau + \int_{t_0}^{t} X_1(t-\tau)D(\tau)y(\tau)dW(\tau) \end{aligned}$$

$$-\int_t^\infty X_2(t-\tau)B(\tau)y(\tau)d\tau - \int_t^\infty X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\,. \quad (3.122)$$

Let every solution $y(t) \equiv y(t,\omega)$ of system (3.114), considered as a random process with the initial condition $y(t_0) = y_0$, correspond to a solution $x(t)$ of system (3.113) with the initial condition

$$x(t_0) = y(t_0) + \int_{t_0}^\infty X_2(t_0-\tau)B(\tau)y(\tau)d\tau + \int_{t_0}^\infty X_2(t_0-\tau)D(\tau)y(\tau)dW(\tau)\,. \tag{3.123}$$

Since the solution $x(t)$ of the linear system (3.113) and the strong solution $y(t) \equiv y(t,\omega)$ of the stochastic differential system (3.114) are defined by the initial conditions, formula (3.123) defines some one-to-one correspondence between the set of solutions $\{y(t) \equiv y(t,\omega)\}$ of system (3.114) and the set of solutions $\{x(t)\}$ of system (3.113).

II) Let us prove that all solutions of (3.114) are mean square bounded. Since

$$y(t) = X(t-t_0)y(t_0) + \int_{t_0}^t X(t-\tau)B(\tau)y(\tau)d\tau + \int_{t_0}^t X(t-\tau)D(\tau)y(\tau)dW(\tau)\,, \tag{3.124}$$

it follows from (3.124), with a use of properties of the stochastic integral, that the square of expectation of $|y(t)|$ satisfies the estimate

$$\begin{aligned}\mathbf{E}|y(t)|^2 &\le 3\|X(t-t_0)\|^2\mathbf{E}|y(t_0)|^2 + 3E\left|\int_{t_0}^t X(t-\tau)B(\tau)y(\tau)d\tau\right|^2 \\ &\quad + 3\mathbf{E}\left|\int_{t_0}^t X(t-\tau)D(\tau)y(\tau)dW(\tau)\right|^2 \le 3\max(a^2,b^2)E|y(t_0)|^2 \\ &\quad + 3\mathbf{E}\left(\int_{t_0}^t \sqrt{\|X(t-\tau)\|}\sqrt{\|X(t-\tau)\|}\sqrt{\|B(\tau)\|}\sqrt{\|B(\tau)\|}|y(\tau)|d\tau\right)^2 \\ &\quad + 3\int_{t_0}^t \mathbf{E}\left|X(t-\tau)D(\tau)y(\tau)\right|^2 d\tau \le 3\max(a^2,b^2)\mathbf{E}|y(t_0)|^2\end{aligned}$$

$$+3\int_{t_0}^{t}\|X(t-\tau)\|\|B(\tau)\|\mathbf{E}|y(\tau)|^2d\tau\int_{t_0}^{t}\|X(t-\tau)\|\|B(\tau)\|d\tau$$

$$+3\int_{t_0}^{t}\|X(t-\tau)\|^2\|D(\tau)\|^2\mathbf{E}|y(\tau)|^2d\tau$$

$$\leq 3\max(a^2,b^2)\left(\mathbf{E}|y(t_0)|^2+\int_0^{\infty}\|B(\tau)\|d\tau\int_{t_0}^{t}\|B(\tau)\|\mathbf{E}|y(\tau)|^2d\tau\right.$$

$$\left.+\int_{t_0}^{t}\|D(\tau)\|^2\mathbf{E}|y(\tau)|^2d\tau\right).$$

Using the Gronwall-Bellman inequality we get

$$\begin{aligned}\mathbf{E}|y(t)|^2 &\leq 3\max(a^2,b^2)\mathbf{E}|y(t_0)|^2e^{3\max(a^2,b^2)\int_{t_0}^{t}(K_1\|B(\tau)\|+\|D(\tau)\|^2)d\tau}\\ &\leq 3\max(a^2,b^2)\mathbf{E}|y(t_0)|^2e^{3\max(a^2,b^2)\int_0^{\infty}(K_1\|B(\tau)\|+\|D(\tau)\|^2)d\tau}\\ &\leq \widetilde{K}\mathbf{E}|y(t_0)|^2\,, \end{aligned}\tag{3.125}$$

where $\widetilde{K}=3\max(a^2,b^2)e^{3\max(a^2,b^2)\int_0^{\infty}(K_1\|B(\tau)\|+\|D(\tau)\|^2)d\tau}$.

Using (3.125) we see that the integrals

$$\int_{t_0}^{\infty}X_2(t_0-\tau)B(\tau)y(\tau)d\tau,\int_{t_0}^{\infty}X_2(t_0-\tau)D(\tau)y(\tau)dW(\tau)$$

converge in mean square.

III) Let us estimate the expectation of the square of the norm of the difference between the corresponding solutions $x(t)$ and $y(t)$. Since

$$x(t)=X(t-t_0)x(t_0)\,,$$

where $x(t_0)$ is defined by formula (3.123), it follows from (3.122) that

$$\mathbf{E}|x(t)-y(t)|^2=\mathbf{E}\left|\int_{t_0}^{t}X_1(t-\tau)B(\tau)y(\tau)d\tau+\int_{t_0}^{t}X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right.$$

$$- \int_{t}^{\infty} X_2(t-\tau)B(\tau)y(\tau)d\tau - \int_{t}^{\infty} X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\Bigg|^2$$

$$\leq 4\mathbf{E}\left|\int_{t_0}^{t} X_1(t-\tau)B(\tau)y(\tau)d\tau\right|^2 + 4\mathbf{E}\left|\int_{t_0}^{t} X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right|^2$$

$$+ 4\mathbf{E}\left|\int_{t}^{\infty} X_2(t-\tau)B(\tau)y(\tau)d\tau\right|^2 + 4\mathbf{E}\left|\int_{t}^{\infty} X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\right|^2 . \tag{3.126}$$

Using (3.125), estimate every term in the right-hand side of inequality (3.126). We have

$$4\mathbf{E}\left|\int_{t_0}^{t} X_1(t-\tau)B(\tau)y(\tau)d\tau\right|^2 \leq 4\mathbf{E}\left(\int_{t_0}^{t} \|X_1(t-\tau)\|\|B(\tau)\||y(\tau)|d\tau\right)^2$$

$$= 4\mathbf{E}\left(\int_{t_0}^{t} \sqrt{\|X_1(t-\tau)\|\|B(\tau)\|}\sqrt{\|X_1(t-\tau)\|\|B(\tau)\|}\,|y(\tau)|d\tau\right)^2$$

$$\leq 4\mathbf{E}\left(\int_{t_0}^{t} \|X_1(t-\tau)\|\|B(\tau)\|d\tau \int_{t_0}^{t} \|X_1(t-\tau)\|\|B(\tau)\|\,|y(\tau)|^2 d\tau\right)$$

$$\leq 4\widetilde{K}\mathbf{E}|y(t_0)|^2\left(\int_{t_0}^{t} \|X_1(t-\tau)\|\|B(\tau)\|d\tau\right)^2$$

$$\leq 4\widetilde{K}\mathbf{E}|y(t_0)|^2\left(\int_{t_0}^{t} ae^{-\alpha(t-\tau)}\|B(\tau)\|d\tau\right)^2 .$$

Since the matrix $B(t)$ is absolutely integrable for $t \geq 2t_0$, we have

$$\int_{t_0}^{t} e^{-\alpha(t-\tau)}\|B(\tau)\|d\tau = \int_{t_0}^{t/2} e^{-\alpha(t-\tau)}\|B(\tau)\|d\tau + \int_{t/2}^{t} e^{-\alpha(t-\tau)}\|B(\tau)\|d\tau$$

$$\le e^{-\frac{\alpha t}{2}}\int\limits_{t_0}^{t/2}\|B(\tau)\|d\tau+\int\limits_{t/2}^{t}\|B(\tau)\|d\tau$$

$$\le e^{-\frac{\alpha t}{2}}\int\limits_{0}^{\infty}\|B(\tau)\|d\tau+\int\limits_{t/2}^{t}\|B(\tau)\|d\tau\,.$$

Since the last expression tends to zero as $t\to\infty$, the first term in relation (3.126) also tends to zero as $t\to\infty$.

Taking into account that the stochastic Ito integral is proper, we get the following estimate for the second term in (3.126):

$$4\mathbf{E}\left|\int\limits_{t_0}^{t}X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right|^2\le 4\int\limits_{t_0}^{t}\mathbf{E}|X_1(t-\tau)D(\tau)y(\tau)|^2d\tau$$

$$\le 4\int\limits_{t_0}^{t}\|X_1(t-\tau)\|^2\|D(\tau)\|^2\mathbf{E}|y(\tau)|^2d\tau$$

$$\le 4\widetilde{K}\mathbf{E}|y(t_0)|^2\int\limits_{t_0}^{t}a^2e^{-2\alpha(t-\tau)}\|D(\tau)\|^2\mathbf{E}|y(\tau)|^2d\tau\,.$$

Now, since the square of the norm of the matrix $D(t)$ is integrable, applying the procedure used to estimate the first term in (3.126), we see that the last term in the above inequality tends to zero as $t\to\infty$.

For the third and the fourth terms in relation (3.126), we have the following inequalities:

$$4\mathbf{E}\left|\int\limits_{t}^{\infty}X_2(t-\tau)B(\tau)y(\tau)d\tau\right|^2\le 4\mathbf{E}\left(\int\limits_{t}^{\infty}\|X_2(t-\tau)\|\|B(\tau)\||y(\tau)|d\tau\right)^2$$

$$\le 4\int\limits_{t}^{\infty}\|X_2(t-\tau)\|\|B(\tau)\|\mathbf{E}|y(\tau)|^2d\tau\int\limits_{t}^{\infty}\|X_2(t-\tau)\|\|B(\tau)\|d\tau$$

$$\le 4\widetilde{K}\mathbf{E}|y(t_0)|^2\left(\int\limits_{t}^{\infty}\|X_2(t-\tau)\|\|B(\tau)\|d\tau\right)^2\le 4\widetilde{K}\mathbf{E}|y(t_0)|^2b^2\left(\int\limits_{t}^{\infty}\|B(\tau)\|d\tau\right)^2$$

and

$$4\mathbf{E}\left|\int_t^\infty X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\right|^2 \le 4\mathbf{E}\int_t^\infty \|X_2(t-\tau)\|^2\|D(\tau)\|^2\mathbf{E}|y(\tau)|^2 d\tau$$

$$\le 4\widetilde{K}\mathbf{E}|y(t_0)|^2 b^2\int_t^\infty \|D(\tau)\|^2 d\tau\,.$$

Using them we see that the last two terms in (3.126) tend to zero as $t \to \infty$. Hence,

$$\lim_{t\to\infty}\mathbf{E}|x(t)-y(t)|^2 = 0\,,$$

which proves the first part of the theorem.

To prove the second part of the theorem, choose a sequence $\{n_k | k \ge 1\}$ such that $n_k > k,\ k \ge 1$, and

$$\int_{n_k}^\infty \|B(\tau)\|d\tau \le \frac{1}{2^k}, \qquad k \ge 1\,,$$

and a sequence $\{m_k | k \ge 1\}$ such that $m_k > k,\ k \ge 1$, and

$$\int_{m_k}^\infty \tau\|D(\tau)\|^2 d\tau \le \frac{1}{2^k}, \qquad k \ge 1\,.$$

Using the sequences n_k and m_k construct a sequence l_k such that

$$l_k = 2\max\{n_k, m_k\}, \qquad k \ge 1\,.$$

Since

$$x(t) = X(t-t_0)x(t_0)\,,$$

where $x(t_0)$ is defined by (3.123), it follows from (3.122) that arbitrary solutions $x(t)$ and $y(t)$ satisfy

$$
\begin{aligned}
&\mathbf{P}\left\{\sup_{t\geq l_k}|x(t)-y(t)|\geq 1/k\right\}\\
&\quad=\mathbf{P}\left\{\sup_{t\geq l_k}\left|\int_{t_0}^{t}X_1(t-\tau)B(\tau)y(\tau)d\tau+\int_{t_0}^{t}X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right.\right.\\
&\qquad\left.\left.-\int_{t}^{\infty}X_2(t-\tau)B(\tau)y(\tau)d\tau-\int_{t}^{\infty}X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\right|\geq 1/k\right\}\\
&\quad\leq\mathbf{P}\left\{\sup_{t\geq l_k}\left|\int_{t_0}^{t}X_1(t-\tau)B(\tau)y(\tau)d\tau\right|\geq\frac{1}{4k}\right\}\\
&\qquad+\mathbf{P}\left\{\sup_{t\geq l_k}\left|\int_{t_0}^{t}X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right|\geq\frac{1}{4k}\right\}\\
&\qquad+\mathbf{P}\left\{\sup_{t\geq l_k}\left|\int_{t}^{\infty}X_2(t-\tau)B(\tau)y(\tau)d\tau\right|\geq\frac{1}{4k}\right\}\\
&\qquad+\mathbf{P}\left\{\sup_{t\geq l_k}\left|\int_{t}^{\infty}X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\right|\geq\frac{1}{4k}\right\},
\end{aligned}
\tag{3.127}
$$

$k\in\mathbf{N}$.

Let us now estimate each term in the above inequality. Using the Chebyshev inequality we find that

$$
\begin{aligned}
&\mathbf{P}\left\{\sup_{t\geq l_k}\left|\int_{t_0}^{t}X_1(t-\tau)B(\tau)y(\tau)d\tau\right|\geq\frac{1}{4k}\right\}\leq 4k\mathbf{E}\sup_{t\geq l_k}\left|\int_{t_0}^{t}X_1(t-\tau)B(\tau)y(\tau)d\tau\right|\\
&\leq 4k\mathbf{E}\sup_{t\geq l_k}\int_{t_0}^{t}\|X_1(t-\tau)\|\|B(\tau)\||y(\tau)|d\tau\\
&\leq 4k\mathbf{E}\sup_{t\geq l_k}\int_{t_0}^{t}ae^{-\alpha(t-\tau)}\|B(\tau)\||y(\tau)|d\tau
\end{aligned}
$$

$$= 4ka\mathbf{E}\sup_{t\ge l_k}\left(\int\limits_{t_0}^{t/2} e^{-\alpha(t-\tau)}\|B(\tau)\||y(\tau)|d\tau + \int\limits_{t/2}^{t} e^{-\alpha(t-\tau)}\|B(\tau)\||y(\tau)|d\tau\right)$$

$$\le 4ka\sqrt{\widetilde{K}\mathbf{E}|y(t_0)|^2}\left(e^{-\frac{\alpha l_k}{2}}\int\limits_{t_0}^{\infty}\|B(\tau)\|d\tau + \int\limits_{l_k/2}^{\infty}\|B(\tau)\|d\tau\right)$$

$$\le 4ka\sqrt{\widetilde{K}\mathbf{E}|y(t_0)|^2}\left(e^{-\frac{\alpha k}{2}}K_1 + \frac{1}{2^k}\right) =: I_k^{(1)}\,.$$

To estimate the second term in inequality (3.127), consider the sequence of random events

$$A_N = \left\{\omega|\sup_{l_k\le t\le N}\left|\int\limits_{t_0}^{t} X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right| \ge \frac{1}{4k}\right\}.$$

It is clear that for arbitrary $N_1 \le N_2$ we have $A_{N_1} \subset A_{N_2}$. Hence, A_N is a monotone sequence of sets, and

$$A = \lim_{N\to\infty} A_N = \bigcup_{N\to\infty} A_N = \left\{\omega|\sup_{l_k\le t}\left|\int\limits_{t_0}^{t} X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right| \ge \frac{1}{4k}\right\},$$

so that

$$\mathbf{P}\{A\} = \lim_{N\to\infty}\mathbf{P}\{A_N\}.$$

It is clear that, for $N \ge l_k$,

$$\sup_{l_k\le t\le N}\left|\int\limits_{t_0}^{t} X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right| \le \sup_{l_k\le t\le N}\left|\int\limits_{t_0}^{l_k} X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right|$$
$$+ \sup_{l_k\le t\le N}\left|\int\limits_{l_k}^{t} X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right|.$$

Thus we have

$$\mathbf{P}\left\{\sup_{l_k\le t\le N}\left|\int\limits_{t_0}^{t} X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right| \ge \frac{1}{4k}\right\}$$

$$\leq \mathbf{P}\left\{\sup_{l_k\leq t\leq N}\left|\int\limits_{t_0}^{l_k} X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right| \geq \frac{1}{8k}\right\}$$

$$+\mathbf{P}\left\{\sup_{l_k\leq t\leq N}\left|\int\limits_{l_k}^{t} X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right| \geq \frac{1}{8k}\right\}. \qquad (3.128)$$

Let us estimate the first term in the right-hand side of inequality (3.128). We have

$$\mathbf{P}\left\{\sup_{l_k\leq t\leq N}\left|\int\limits_{t_0}^{l_k} X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right| \geq \frac{1}{8k}\right\}$$

$$\leq 64k^2\mathbf{E}\left(\sup_{l_k\leq t\leq N}\left|\int\limits_{t_0}^{l_k} X_1(t-\tau)D(\tau)y(\tau)|^2dW(\tau)\right|^2\right)$$

$$\leq 64k^2\widetilde{K}\mathbf{E}|y(t_0)|^2\left(e^{-\alpha l_k}\int\limits_{t_0}^{l_k/2}\|D(\tau)\|^2d\tau + \int\limits_{l_k/2}^{l_k}\|D(\tau)\|^2d\tau\right)$$

$$\leq 64k^2\widetilde{K}\mathbf{E}|y(t_0)|^2\left(e^{-\alpha k}\int\limits_{t_0}^{\infty}\|D(\tau)\|^2d\tau + \int\limits_{l_k/2}^{\infty}\|D(\tau)\|^2d\tau\right)$$

$$\leq 64k^2\widetilde{K}\mathbf{E}|y(t_0)|^2\left(e^{-\alpha k}K_1 + \frac{1}{2^k}\right) =: I_k^{(2)}.$$

For the second term in the right-hand side of (3.128), we have

$$\mathbf{P}\left\{\sup_{l_k\leq t\leq N}\left|\int\limits_{l_k}^{t} X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right| \geq \frac{1}{8k}\right\}$$

$$= \mathbf{P}\left\{\sup_{l_k\leq t\leq N}\left|\int\limits_{l_k}^{t}(X_1(t-\tau)-X_1(t-l_k)+X_1(t-l_k))D(\tau)y(\tau)dW(\tau)\right| \geq \frac{1}{8k}\right\}$$

$$\leq \mathbf{P}\left\{\sup_{l_k\leq t\leq N}\left|\int\limits_{l_k}^{t}(X_1(t-\tau)-X_1(t-l_k))D(\tau)y(\tau)dW(\tau)\right| \geq \frac{1}{16k}\right\}$$

$$+\mathbf{P}\left\{\sup_{l_k\le t\le N}\left|\int\limits_{l_k}^{t} X_1(t-l_k)D(\tau)y(\tau)dW(\tau)\right|\ge\frac{1}{16k}\right\}. \tag{3.129}$$

Find estimates for each term in the above inequality:

$$\begin{aligned}
&\mathbf{P}\left\{\sup_{l_k\le t\le N}\left|\int\limits_{l_k}^{t} X_1(t-l_k)D(\tau)y(\tau)dW(\tau)\right|\ge\frac{1}{16k}\right\}\\
&\quad\le\mathbf{P}\left\{\sup_{l_k\le t\le N}\|X_1(t-l_k)\|\sup_{l_k\le t\le N}\left|\int\limits_{l_k}^{t} D(\tau)y(\tau)dW(\tau)\right|\ge\frac{1}{16k}\right\}\\
&\quad\le 256k^2a^2\int\limits_{l_k}^{N}||D(\tau)||^2\tilde{K}E|y(t_0)|^2d\tau\\
&\quad=\tilde{K}E|y(t_0)|^2 256k^2a^2\int\limits_{l_k}^{\infty}||D(\tau)||^2\tau d\tau\\
&\quad\le\widetilde{K}E|y(t_0)|^2 256k^2a^2\frac{1}{2^k}.
\end{aligned} \tag{3.130}$$

To estimate the other term in the right-hand side of inequality (3.129), consider the following:

$$\begin{aligned}
&\int_{l_k}^{t}(X_1(t-\tau)-X_1(t-l_k))D(\tau)y(\tau)dW(\tau)\\
&\quad=-\int_{l_k}^{t}\left(\int_{l_k}^{\tau}X_1(t-s)Ads\right)D(\tau)y(\tau)dW(\tau)\\
&\quad=-\int_{l_k}^{t}\left(\int_{l_k}^{t}X_1(t-s)AI_{\{s\le\tau\}}ds\right)D(\tau)y(\tau)dW(\tau)\\
&\quad=-\int_{l_k}^{t}X_1(t-s)A\left(\int_{l_k}^{t}I_{\{s\le\tau\}}D(\tau)y(\tau)dW(\tau)\right)ds\,.
\end{aligned}$$

This gives

$$\begin{aligned}
&\mathbf{P}\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^{t}(X_1(t-\tau)-X_1(t-l_k))D(\tau)y(\tau)dW(\tau)\right|\ge\frac{1}{16k}\right\}\\
&=\mathbf{P}\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^{t}X_1(t-s)A\int_{l_k}^{t}I_{\{s\le\tau\}}D(\tau)y(\tau)dW(\tau)ds\right|\ge\frac{1}{16k}\right\}
\end{aligned}$$

$$\leq 256k^2E\left(\sup_{l_k\leq t\leq N}\left|\int_{l_k}^t X_1(t-s)A\left(\int_{l_k}^t I_{\{s\leq\tau\}}D(\tau)y(\tau)dW(\tau)\right)ds\right|\right)^2$$

$$\leq 256k^2E\sup_{l_k\leq t\leq N}\left(\int_{l_k}^t ae^{-\alpha(t-s)}\|A\|\left|\int_{l_k}^t I_{\{s\leq\tau\}}D(\tau)y(\tau)dW(\tau)\right|ds\right)^2$$

$$\leq 256k^2E\left(\sup_{l_k\leq t\leq N}\left(\int_{l_k}^t a^2e^{-2\alpha(t-s)}\|A\|^2ds\int_{l_k}^t\left|\int_{l_k}^t I_{s\leq\tau}D(\tau)y(\tau)dW(\tau)\right|^2ds\right)\right)$$

$$\leq \frac{256k^2a^2\|A\|^2}{2\alpha}E\left(\sup_{l_k\leq t\leq N}\int_{l_k}^t\left|\int_{l_k}^t I_{\{s\leq\tau\}}D(\tau)y(\tau)dW(\tau)\right|^2ds\right)$$

$$\leq \frac{256k^2a^2\|A\|^2}{2\alpha}\int_{l_k}^N E\sup_{l_k\leq t\leq N}\left|\int_{l_k}^t I_{\{s\leq\tau\}}D(\tau)y(\tau)dW(\tau)\right|^2ds$$

$$\leq \frac{1024k^2a^2\|A\|^2}{2\alpha}\int_{l_k}^N\left(\int_s^N D^2(\tau)E|y(\tau)|^2d\tau|ds\right)$$

$$\leq \frac{\widetilde{K}E|y(t_0)|^2 1024a^2\|A\|^2}{2\alpha}k^2\int_{l_k}^N\left(\int_s^N D^2(\tau)d\tau\right)ds)$$

$$= \frac{\widetilde{K}E|y(t_0)|^2 1024a^2\|A\|^2}{2\alpha}k^2\int_{l_k}^N\left(D^2(\tau)\int_{l_k}^\tau ds\right)d\tau)$$

$$\leq \frac{\widetilde{K}E|y(t_0)|^2 1024a^2\|A\|^2}{2\alpha}k^2\int_{l_k}^N \tau D^2(\tau)d\tau$$

$$\leq \frac{\widetilde{k}E|y(t_0)|^2 1024a^2\|A\|^2}{2\alpha}k^2 2^{-k}\,. \tag{3.131}$$

It follows now from (3.130) and (3.131) that

$$\mathbf{P}\left\{\sup_{l_k\leq t\leq N}\left|\int_{l_k}^t X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right|\geq\frac{1}{8k}\right\}$$

$$\leq \tilde{K}E|y(t_0)|^2 256k^2(1+4a^2\|A\|^2)2^{-k} =: I_k^{(3)}\,.$$

By making N tend to infinity, we get from (3.128) that

$$\mathbf{P}\left\{\sup_{l_k\leq t}\left|\int_{t_0}^t X_1(t-\tau)D(\tau)y(\tau)dW(\tau)\right|\geq\frac{1}{4k}\right\}\leq I_k^{(2)}+I_k^{(3)}.$$

Let us now estimate the third term in the right-hand side of inequality (3.127). We get

$$\mathbf{P}\left\{\sup_{t\geq l_k}\left|\int_t^\infty X_2(t-\tau)B(\tau)y(\tau)d\tau\right|\geq\frac{1}{4k}\right\}$$

$$\leq \mathbf{P}\left\{\sup_{t\geq l_k}\int_t^\infty \|X_2(t-\tau)\|\|B(\tau)\||y(\tau)|d\tau\geq\frac{1}{4k}\right\}$$

$$\leq \mathbf{P}\left\{\sup_{t\geq l_k}\int_{l_k}^\infty \|X_2(t-\tau)\|\|B(\tau)\||y(\tau)|d\tau\geq\frac{1}{4k}\right\}$$

$$\leq 4k\mathbf{E}\sup_{t\geq l_k}\int_{l_k}^\infty \|X_2(t-\tau)\|\|B(\tau)\||y(\tau)|d\tau$$

$$\leq 4kb\sqrt{\widetilde{K}\mathbf{E}|y(t_0)|^2}\int_{l_k}^\infty\|B(\tau)\|d\tau\leq 4kb\sqrt{\widetilde{K}\mathbf{E}|y(t_0)|^2}\,\frac{1}{2^k}=:I_k^{(4)}.$$

Finally, estimate the last term in the right-hand side of inequality (3.127). To this end, as above we consider the sequence of random events,

$$A_N=\left\{\omega|\sup_{l_k\leq t\leq N}\left|\int_t^\infty X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\right|\geq\frac{1}{4k}\right\}.$$

Since A_N is a monotone sequence of sets, we have

$$A=\lim_{N\to\infty}A_N=\bigcup_{N\to\infty}A_N=\left\{\omega|\sup_{l_k\leq t}\left|\int_t^\infty X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\right|\geq\frac{1}{4k}\right\},$$

and

$$\mathbf{P}\{A\}=\lim_{N\to\infty}\mathbf{P}\{A_N\}.$$

It is clear that, for $t \geq l_k$,

$$\begin{aligned}&\int_t^\infty X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\\&\quad=\int_{l_k}^\infty X_2(t-\tau)D(\tau)y(\tau)dW(\tau)-\int_{l_k}^t X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\,,\end{aligned}$$

and, hence,

$$\begin{aligned}&\mathbf{P}\left\{\sup_{l_k\leq t\leq N}\left|\int_t^\infty X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\right|\geq\frac{1}{4k}\right\}\\&\quad\leq\mathbf{P}\left\{\sup_{l_k\leq t\leq N}\left|\int_{l_k}^\infty X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\right|\geq\frac{1}{8k}\right\}\\&\qquad+\mathbf{P}\left\{\sup_{l_k\leq t\leq N}\left|\int_{l_k}^t X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\right|\geq\frac{1}{8k}\right\}.\end{aligned}\tag{3.132}$$

Estimate each term in the above inequality. We have

$$\begin{aligned}&\mathbf{P}\left\{\sup_{l_k\leq t\leq N}\left|\int_{l_k}^\infty X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\right|\geq\frac{1}{8k}\right\}\\&\quad\leq\mathbf{P}\left\{\sup_{l_k\leq t\leq N}\|X_2(t)\|\left|\int_{l_k}^\infty X_2^{-1}(\tau)D(\tau)y(\tau)dW(\tau)\right|\geq\frac{1}{8k}\right\}\\&\quad\leq 64b^2k^2\mathbf{E}|\left|\int_{l_k}^\infty X_2^{-1}(\tau)D(\tau)y(\tau)dW(\tau)\right|^2\leq 64b^4k^2\widetilde{K}\mathbf{E}|y(t_0)|^2\int_{l_k}^\infty\|D(\tau)\|^2d\tau\\&\quad\leq 64b^4k^2\widetilde{K}\mathbf{E}|y(t_0)|^2\frac{1}{2^k}=:I_k^{(5)}\,.\end{aligned}$$

Now we have

$$\mathbf{P}\left\{\sup_{l_k\leq t\leq N}\left|\int_{l_k}^t X_2(t-\tau)D(\tau)y(\tau)dW(\tau)\right|\geq\frac{1}{8k}\right\}$$

$$\leq 64k^2b^4 \int\limits_{l_k}^{N} \|D(\tau)\|^2 \mathbf{E}|y(\tau)|^2 d\tau \leq 64k^2b^4 \widetilde{K} \mathbf{E}|y(t_0)|^2 \frac{1}{2^k} =: I_k^{(6)} .$$

By making $N \to \infty$ in (3.132), we get

$$\mathbf{P}\left\{ \sup_{l_k \leq t} \left| \int\limits_{t}^{\infty} X_2(t-\tau)D(\tau)y(\tau)dW(\tau) \right| \geq \frac{1}{4k} \right\} \leq I_k^{(5)} + I_k^{(6)} ,$$

or, finally,

$$\mathbf{P}\left\{ \sup_{t \geq l_k} |x(t) - y(t)| \geq 1/k \right\} \leq \sum_{j=1}^{5} I_k^{(j)} = I_k .$$

It is clear that the series $\sum\limits_{k=1}^{\infty} I_k$ is convergent and, hence, it follows from the Borel-Cantelli lemma that there exists a positive integer $M = M(\omega)$ such that, for arbitrary $k \geq M(\omega)$,

$$\sup_{t \geq l_k} |x(t) - y(t)| < 1/k \tag{3.133}$$

with probability 1.

Thus, for almost all ω and arbitrary $\varepsilon > 0$ there exists $T = T(\varepsilon, \omega) = l_{k_0}$, where $k_0 = \max\{[1/\varepsilon], M(\omega)\}$, such that the following inequality holds for all $t \geq T$:

$$|x(t) - y(t)| \leq \sup_{t \geq T} |x(t) - y(t)| \leq 1/k_0 \leq \varepsilon ,$$

which finishes the proof of the theorem. □

Remark 1. It follows from the proof of the preceding theorem that if all solutions of system (3.113) are bounded on the axis, then condition (3.116) can be replaced with the condition

$$\int_0^\infty \|D(t)\|^2 dt \leq K_1 < \infty . \tag{3.134}$$

Remark 2. Let us remark that the correspondence between solutions of systems (3.113) and (3.114), constructed in the proof of the theorem, is not one-to-one as opposed to the Levinson theorem. This has to do with the fact that the set of solutions of system (3.113) is larger than the set of solutions of system (3.114), — the latter must be F_t-measurable. This imposes an additional measurability condition on the initial conditions for solution of system (3.114).

Such restrictions on the initial conditions for solutions of system (3.113) are not needed, since the system is deterministic with constant coefficients. Thus, with the correspondence constructed in Theorem 3.10, the deterministic system (3.113) reconstructs only coarse properties of the stochastic system (3.114), namely, the boundedness of solutions and their stability in some sense, dissipativity, etc. However, a correspondence between solutions of systems (3.114) and (3.113) can be constructed in such a way that a nontrivial solution of system (3.114) corresponds to a nontrivial solution of system (3.113).

To prove the following theorem, we will need two well-known theorems from the theory of linear systems.

Lemma 3.9. *An arbitrary solution $x(t)$ of system* (3.113) *can be written as*

$$x(t) = X(t,\tau)x(\tau)\,,$$

where $X(t,\tau)$ is a matriciant of system (3.113)*, $X(\tau,\tau) = E$.*

Lemma 3.10. *An arbitrary solution $y(t, y_0)$ of system* (3.114) *can be written as*

$$y(t, y_0) = Y(t)y_0\,, \tag{3.135}$$

where $Y(t,\omega)$ is a fundamental matrix of system (3.114)*, nondegenerate with probability* 1 *for every $t \geq 0$, $Y(0) = E$.*

Theorem 3.11. *Let conditions of Theorem* 3.10 *be satisfied. Then the correspondence that defines asymptotic equivalence between solutions of systems* (3.114) *and* (3.113) *can be constructed in such a way that every nontrivial solution of system* (3.114) *corresponds to a nontrivial solution of system* (3.113).

Proof. Note that relations (3.135) hold and

$$\det Y(t) \neq 0 \tag{3.136}$$

for every t with probability 1. However, since $Y(t)$ and $y(t, y_0)$ are continuous with probability 1 there exists a set $Z \subset \Omega$, $\mathbf{P}(Z) = 1$, such that, for arbitrary $\omega \in Z$, relations (3.135) and (3.136) hold for all $t \geq 0$. Denote $Z_0 = \overline{Z}$.

Take now an arbitrary nonzero solution $y(t, y_0)$ of system (3.114). Denote by $A_0 \subset \Omega$ the set

$$A_0 = \{\omega : |y_0(\omega)| \neq 0\}.$$

It is clear that $\mathbf{P}(A_0) = 1$. Let $A_1 = A_0 \backslash Z_0$. Then, for arbitrary $\omega \in A_1$, we see that

1) relations (3.135) and (3.136) hold for all $t \geq 0$;

2) $|y(t, y_0)| \neq 0$ for arbitrary $t \geq 0$.

When proving Theorem 3.10, the considered correspondence between solutions of systems (3.114) and (3.113) was constructed via formula (3.123). Using (3.135) we can rewrite (3.123) as

$$x(t_0) = (E + \int_{t_0}^{\infty} X_2(t_0 - \tau)B(\tau)Y(\tau)d\tau + \int_{t_0}^{\infty} X_2(t_0 - \tau)D(\tau)Y(\tau)dW(\tau))y_0$$

or

$$x(t_0) = [E + \Phi(t_0, \omega)]y_0(\omega). \tag{3.137}$$

If we could prove that $||\Phi(t_0, \omega)|| < 1$ with probability 1, then relation (3.137) would imply that $x(t_0)$ is a nonzero vector. It was shown in the proof of Theorem 3.10 that

$$\Phi(t_0, \omega) \to 0 \tag{3.138}$$

with probability 1 as $t_0 \to \infty$.

From the set A_1, remove those $\omega \in \Omega$ for which (3.138) does not hold. Denote the obtained set by A_2. Evidently, $\mathbf{P}(A_2) = 1$. For every $\omega \in A_2$ there exists an integer $t_0(\omega)$ such that

$$||\Phi(t_0, \omega)|| < 1\,. \tag{3.139}$$

Hence,

$$\det(E + \Phi(t_0(\omega), \omega)) \neq 0\,.$$

Consider such $\omega \in A_2$. For $\omega \in A_2$, we have the following:

1) $|y_0(\omega)| \neq 0$;

2) there is an integer $t_0(\omega)$ such that $\det(E + \Phi(t_0(\omega), \omega)) \neq 0$;

3) $y(t, y_0(\omega), \omega)$ is nonzero for every $t \geq 0$;

4) formula (3.135) holds for every $t \geq 0$.

By Theorem 3.10, for every $t_0 > 0$ there is a correspondence between a solution $y(t, y_0)$ of system (3.114) and a solution $x(t)$ of system (3.113) with the initial condition $x(t_0)$ defined by formula (3.137) such that

$$|y(t, y_0) - x(t)| \to 0, \qquad t \to \infty\,,$$

with probability 1 for every t_0.

Take t_0 to be an integer. For every such t_0 there is a correspondence between a solution $y(t, y_0)$ of system (3.114) and a solution $x(t, t_0)$ of system (3.113) with the initial condition $x(t_0, t_0) = x(t_0)$ defined by formula (3.137). Then

$$|y(t, y_0) - x(t, t_0)| \to 0, \qquad t \to \infty\,,$$

with probability 1.

Denote by $C_{t_0} \subset \Omega$ the set of ω for which the boundary-value condition is violated. Let

$$C = \cup_{t_0} C_{t_0}\,.$$

It is clear that $\mathbf{P}(C) = 0$. Consider the set $A_3 = A_2 \backslash C$. We will now consider $\omega \in A_3$. For such ω, the following properties are satisfied:

1) $|y_0(\omega)| \neq 0\,;$

2) there is an integer $t_0(\omega)$ such that $\det(E + \Phi(t_0(\omega), \omega)) \neq 0$;

3) $y(t, y_0(\omega), \omega)$ is nonzero for every $t \geq 0$;

4) formula (3.135) holds for every $t \geq 0$;

5) for every integer $t_0 > 0$,

$$|y(t, y_0, \omega) - x(t, t_0, \omega)| \to 0, \qquad t \to \infty,$$

with probability 1.

Take now an arbitrary $\omega_0 \in A_3$ and look at the trajectory $y(t, y_0(\omega_0), \omega_0)$. For such ω_0 there exists an integer $t_0(\omega_0)$ such that

$$\det(E + \Phi(t_0(\omega_0), \omega_0)) \neq 0\,.$$

Set now $t_0(\omega_0) = t_1$ and keep it fixed for all ω. To every solution $y(t, y_0)$ of system (3.114), for a given point t_1, put $x(t, t_1)$ into correspondence by formula (3.137) such that

$$|y(t, y_0) - x(t, t_1)| \to 0, \qquad t \to \infty$$

with probability 1. This relation also holds for ω_0. The value of $x(t, t_1)$ at the point t_1 is defined for all $\omega \in A_3$, in particular, for ω_0. But then, by Lemma 3.9, for all $t \geq 0$, we have

$$x(t, t_1) = X(t, t_1)x(t_1)\,,$$

in particular, for ω_0,

$$x(t_1, t_1, \omega_0) = x(t_0(\omega_0), t_0(\omega_0), \omega_0) \,.$$

Using properties of the set A_3 we see that $y(t_0(\omega_0), y_0(\omega_0), \omega_0)$ is nonzero. Then

$$x(t_0(\omega_0), t_0(\omega_0), \omega_0) = (E + \Phi(t_0(\omega_0), \omega_0))y(t_0(\omega_0), y_0(\omega_0), \omega_0)$$

is also nonzero. Let now

$$x_0(\omega_0) = x(0, t_0(\omega_0)) = X(0, t_1)x(t_0(\omega_0), t_0(\omega_0), \omega_0) \,. \tag{3.140}$$

Since the matrix $X(t, \tau)$ is nondegenerate, $x_0(\omega_0)$ is nonzero.

This procedure is now applied for every $\omega_0 \in A_3$. As a result we have a solution $y(t, y_0(\omega_0), \omega_0)$ with the initial condition $y_0(\omega_0)$ in zero corresponding to a solution $x(t, x_0(\omega_0)) = x_0(\omega_0)$ of system (3.113) with the initial conditions in zero being

$$x(0, x_0(\omega_0)) = x_0(\omega_0) \,.$$

Such a solution is nonzero for every $\omega_0 \in A_3$. For other $\omega \in \Omega \backslash A_3$, the set of which has measure zero, the correspondence $y_0(\omega) \to x_0(\omega)$ is constructed in an arbitrary way.

So, to every nontrivial solution $y(t, y_0(\omega), \omega)$ of system (3.114) there corresponds a nontrivial solution $x(t, x_0(\omega))$ of system (3.113) for all ω.

It remains to show that

$$|y(t, y_0) - x(t, x_0(\omega))| \to 0, \qquad t \to \infty \,. \tag{3.141}$$

with probability 1.

Indeed, take an arbitrary $\omega_0 \in A_3$. Use the above procedure to define $x_0(\omega_0)$. Consider now the trajectories of the solutions $y(t, y_0(\omega_0), \omega_0)$ and $x(t, x_0(\omega_0))$. There is an integer $t_0(\omega_0)$ for ω_0 such that the nondegeneracy condition $\det(E + \Phi(t_0(\omega_0), \omega_0) \neq 0$ holds true. Set $t_0(\omega_0) = t_1$ and let it be fixed. Consider the solution $y(t, y_0(\omega))$ at the point t_1. For every positive t_1, by Lemma 3.10, we have the identity

$$y(t_1, y_0(\omega_0), \omega_0) = Y(t_1, \omega_0)y_0(\omega_0) \,.$$

Hence,

$$y(t_0(\omega_0), y_0(\omega_0), \omega_0) = Y(t_0(\omega_0), \omega_0)y_0(\omega_0) \,,$$

and, since $t_0(\omega_0) = t_1$, we have that $Y(t_0(\omega_0), \omega_0) = Y(t_1, \omega_0)$. This implies that

$$y(t_0(\omega_0), y_0(\omega_0), \omega_0) = y(t_1, y_0(\omega_0), \omega_0)\,.$$

In the same way, one can prove that

$$x(t_1, x_0(\omega_0), \omega_0) = x(t_0(\omega_0), x_0(\omega_0), \omega_0)\,.$$

It follows from the above that

$$|y(t, y_0(\omega_0), \omega_0) - x(t, x_0(\omega_0), \omega_0)| \to 0, \qquad t \to \infty\,.$$

Since ω_0 is arbitrary, this finishes the proof of the theorem. □

Let us give an application example for Theorem 3.10.

Example 1. Consider a system of ordinary differential equations,

$$\begin{aligned} dx_1 &= -x_1 dt, \\ dx_2 &= (x_1 - x_2)dt. \end{aligned} \tag{3.142}$$

It is easy to see that all solutions of this system are bounded on the positive semiaxis. Together with system (3.142), consider the following system of stochastic differential equations:

$$\begin{aligned} dy_1 &= \left(-1 + \frac{\sqrt{2}}{(t+1)^3} \right) y_1 dt + \frac{1}{(t+1)^2} y_2 dW(t)\,, \\ dy_2 &= \left(y_1 + \left(-1 + \frac{\sqrt{2}}{(t+1)^3} \right) y_2 \right) dt + \frac{1}{(t+1)^2} y_1 dW(t)\,. \end{aligned} \tag{3.143}$$

Let us use Theorem 3.10 to investigate the asymptotic behavior of solutions of system (3.143). It is clear that

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix},\ B(t) = \begin{pmatrix} \dfrac{\sqrt{2}}{(t+1)^3} & 0 \\ 0 & \dfrac{\sqrt{2}}{(t+1)^3} \end{pmatrix},\ D(t) = \begin{pmatrix} 0 & \dfrac{1}{(t+1)^2} \\ \dfrac{1}{(t+1)^2} & 0 \end{pmatrix}.$$

Let us calculate $\|B(t)\|$ and $\|D(t)\|^2$. We have

$$\|B(t)\| = \frac{2}{(t+1)^3}, \qquad \|D(t)\|^2 = \frac{2}{(t+1)^4}.$$

Then
$$\int_0^\infty \|B(t)\|dt < \infty, \qquad \int_0^\infty t\|D(t)\|^2 dt < \infty.$$

Hence, the theorem implies that system (3.143) is asymptotically equivalent to system (3.142) in mean square and with probability 1.

Corollary 3.2. *It follows from the conditions of Theorem* 3.10 *that stability of system* (3.113) *implies that system* (3.114) *is stable in mean square and with probability* 1,

Proof. I. To prove the latter fact, it is sufficient to show that the dependence $x_0 = F(y_0, \omega)$ given by relation (3.123) is continuous in a certain probabilistic sense. The latter follows from the estimate

$$\mathbf{E}|y(t,y_0) - y(t,y_1)|^2 \le 3(\mathbf{E}|y(t,y_0) - x(t,x_0)|^2 + \mathbf{E}|x(t,x_0) - x(t,x_1)|^2 + \mathbf{E}|x(t,x_1) - y(t,y_1)|^2)\,. \tag{3.144}$$

Since the systems are asymptotically equivalent, for arbitrary $\varepsilon > 0$, by choosing a sufficiently large $T > 0$, the first and the third terms in the above inequality can be made less than ε for $t > T$. The choice for a sufficiently small $\delta > 0$ in the inequality

$$\mathbf{E}|y_0 - y_1|^2 < \delta$$

will make the second term in (3.144) small, as well as the difference $\mathbf{E}|y(t,y_0) - y(t,y_1)|^2$ for $t \in [0,T]$.

In the same way, one can show that stability of system (3.113) and continuity with probability 1 of the mapping $x_0 = F(y_0,\omega)$ implies stability of system (3.114) with probability 1.

II. Let us prove that the mappings $x_0 = F(y_0,\omega)$ are continuous with probability 1.

Let $Y(t,t_0)$ me a matriciant of system (3.114). Then, as is well known, every solution $y(t,y_0)$ such that $y(t_0,y_0) = y_0$ can be written as

$$y(t,y_0) = Y(t,t_0)y_0,$$

and the mappings $x_0 = F(y_0,\omega)$ will take the form

$$x_0 = y_0 + \int_{t_0}^{\infty} X_2(t_0 - \tau)B(\tau)Y(\tau,t_0)y_0 d\tau$$

$$+ \int_{t_0}^{\infty} X_2(t_0 - \tau) D(\tau) Y(\tau, t_0) y_0 dW(\tau) . \tag{3.145}$$

Choose an arbitrary sequence $\{y_n\}$ such that $y_n \to y_0, n \to \infty$, with probability 1, that is, such that

$$\mathbf{P}\{\lim_{n\to\infty} |y_n - y_0| > 0\} = 0 .$$

Consider the quantity $\mathbf{P}\{\lim_{n\to\infty} |x_n - x_0| > 0\}$, where $x_n = F(y_n, \omega)$. We have

$$\begin{aligned}
&\mathbf{P}\{\lim_{n\to\infty} |x_n - x_0| > 0\} \\
&\quad \le \mathbf{P}\left\{\lim_{n\to\infty} \left(|y_n - y_0| + \left| \int_{t_0}^{\infty} X_2(t_0 - \tau) B(\tau) Y(\tau, t_0)(y_n - y_0) d\tau \right| \right.\right. \\
&\qquad \left.\left. + \left| \int_{t_0}^{\infty} X_2(t_0 - \tau) D(\tau) Y(\tau, t_0)(y_n - y_0) dW(\tau) \right| \right) > 0 \right\} \\
&\quad \le \mathbf{P}\left\{\lim_{n\to\infty} |y_n - y_0| > 0\right\} \\
&\qquad + \mathbf{P}\left\{\lim_{n\to\infty} \left| \int_{t_0}^{\infty} X_2(t_0 - \tau) B(\tau) Y(\tau, t_0)(y_n - y_0) d\tau \right| > 0 \right\} \\
&\qquad + \mathbf{P}\left\{\lim_{n\to\infty} \left| \int_{t_0}^{\infty} X_2(t_0 - \tau) D(\tau) Y(\tau, t_0)(y_n - y_0) dW(\tau) \right| > 0 \right\} .
\end{aligned}$$

Let us estimate the two last terms in the right-hand side of the inequality. We have

$$\begin{aligned}
&\mathbf{P}\left\{\lim_{n\to\infty} \left| \int_{t_0}^{\infty} X_2(t_0 - \tau) B(\tau) Y(\tau, t_0)(y_n - y_0) d\tau \right| > 0 \right\} \\
&\quad \le \mathbf{P}\left\{\lim_{n\to\infty} \int_{t_0}^{\infty} \|X_2(t_0 - \tau)\| \|B(\tau)\| \|Y(\tau, t_0)\| d\tau \cdot |y_n - y_0| > 0 \right\} .
\end{aligned}$$

Let us prove that the quantity $\int_{t_0}^{\infty} \|X_2(t_0 - \tau)\| \|B(\tau)\| \|Y(\tau, t_0)\| d\tau$ is finite with probability 1. Since it is positive, it is sufficient to show that it is mean square bounded,

$$\mathbf{E}\left(\int_{t_0}^{\infty} \|X_2(t_0-\tau)\|\|B(\tau)\|\|Y(\tau,t_0)\|d\tau\right)^2$$

$$\leq \int_{t_0}^{\infty} \|X_2(t_0-\tau)\|\|B(\tau)\|d\tau \int_{t_0}^{\infty} \|X_2(t_0-\tau)\|\|B(\tau)\|\mathbf{E}\|Y(\tau,t_0)\|^2 d\tau \leq C_1\,,$$

where C_1 is a positive constant. Hence, there exists a positive random variable $\widetilde{C}_1$ such that

$$\mathbf{P}\left\{\int_{t_0}^{\infty} \|X_2(t_0-\tau)\|\|B(\tau)\|\|Y(\tau,t_0)\|d\tau \leq \widetilde{C}_1\right\} = 1\,. \tag{3.146}$$

Now we have

$$\mathbf{P}\left\{\lim_{n\to\infty}\left|\int_{t_0}^{\infty} X_2(t_0-\tau)D(\tau)Y(\tau,t_0)(y_n-y_0)dW(\tau)\right| > 0\right\}$$

$$\leq \mathbf{P}\left\{\lim_{n\to\infty}\left\|\int_{t_0}^{\infty} X_2(t_0-\tau)D(\tau)Y(\tau,t_0)(y_n-y_0)dW(\tau)\right\| |y_n-y_0| > 0\right\}.$$

Let us prove that

$$\left\|\int_{t_0}^{\infty} X_2(t_0-\tau)D(\tau)Y(\tau,t_0)dW(\tau)\right\|$$

is finite with probability 1 by estimating the square of its expectation. So,

$$\mathbf{E}\left\|\int_{t_0}^{\infty} X_2(t_0-\tau)D(\tau)Y(\tau,t_0)\frac{y_n-y_0}{|y_n-y_0|}dW(\tau)\right\|^2$$

$$\leq \int_{t_0}^{\infty} \|X_2(t_0-\tau)\|^2\|D(\tau)\|^2\mathbf{E}\|Y(\tau,t_0)\|^2 d\tau \leq C_2,$$

where C_2 is a positive constant. Hence, there is a positive random constant $\widetilde{C}_2$ such that

$$\mathbf{P}\left\{\left\|\int_{t_0}^{\infty} X_2(t_0-\tau)D(\tau)Y(\tau,t_0)dW(\tau)\right\| \leq \widetilde{C}_2\right\} = 1\,, \tag{3.147}$$

and, hence,

$$\mathbf{P}\{\lim_{n\to\infty}|x_n - x_0| > 0\} \le \mathbf{P}\{\lim_{n\to\infty}|y_n - y_0| > 0\}$$
$$+\mathbf{P}\{\widetilde{C}_1 \lim_{n\to\infty}|y_n - y_0| > 0\} + \mathbf{P}\{\widetilde{C}_2 \lim_{n\to\infty}|y_n - y_0| > 0\} = 0.$$

III. Let us prove that the mapping $x_0 = F(y_0, \omega)$ is mean square continuous. Choose an arbitrary sequence $\{y_n\}$ such that $y_n \to y_0$ for $n \to \infty$ in mean square. Consider $\mathbf{E}|x_n - x_0|^2$, where $x_n = F(y_n, \omega)$. We have

$$\mathbf{E}|x_n - x_0|^2 \le 3\mathbf{E}|y_n - y_0|^2$$
$$+\,3\mathbf{E}\left|\int_{t_0}^{\infty} X_2(t_0-\tau)B(\tau)Y(\tau,t_0)(y_n - y_0)d\tau\right|^2$$
$$+\,3\mathbf{E}\left|\int_{t_0}^{\infty} X_2(t_0-\tau)D(\tau)Y(\tau,t_0)(y_n - y_0)dW(\tau)\right|^2 .$$

Using inequality (3.146) we get

$$\mathbf{E}\left|\int_{t_0}^{\infty} X_2(t_0-\tau)B(\tau)Y(\tau,t_0)(y_n - y_0)d\tau\right|^2$$
$$\mathbf{E}\left(|y_n - y_0|\int_{t_0}^{\infty}\|X_2(t_0-\tau)\|\|B(\tau)\|\|Y(\tau,t_0)\|d\tau\right)^2 \le C_1\mathbf{E}|y_n - y_0|^2 .$$

In the same way, inequality (3.147) implies that

$$\mathbf{E}\left|\int_{t_0}^{\infty} X_2(t_0-\tau)D(\tau)Y(\tau,t_0)(y_n - y_0)dW(\tau)\right|^2 \le C_2\mathbf{E}|y_n - y_0|^2 .$$

Hence,

$$\lim_{n\to\infty}\mathbf{E}|x_n - x_0|^2 \le 3(1 + C_1 + C_2)\lim_{n\to\infty}\mathbf{E}|y_n - y_0|^2 = 0\,,$$

which proves the corollary. □

Consider the system

$$dy = B(t)ydt + D(t)ydW(t)\,. \tag{3.148}$$

We have the following.

Corollary 3.3. *Let estimates* (3.115) *hold. Then every solution $y = y(t, \omega)$ of system* (3.148), *with probability* 1, *has a horizontal asymptote $y = C$, which is random in general, $y = C(\omega)$, in the sense that*

$$P\left\{\lim_{t\to\infty} |y(t,\omega) - C(\omega)| = 0\right\} = 1\,.$$

Consider the following stochastic second order differential equation:

$$\frac{d^2y}{dt^2} + \left[a^2 + b(t)\frac{dW(t)}{dt}\right]y = 0\,, \tag{3.149}$$

where $b(t)$ is a function continuous on the positive semiaxis and such that $\int\limits_0^\infty b^2(\tau)d\tau < \infty$, and $W(t)$ is a Wiener process. Since a Wiener process does not have a derivative, equation (3.149) is formal. To give a rigorous meaning to (3.149), rewrite it as a system of stochastic Ito equations,

$$\begin{aligned} dy_1 &= y_2 dt, \\ dy_2 &= -a^2 y_1 dt - b(t) y_1 dW(t). \end{aligned}$$

Theorem 3.10 and Corollary 3.2 easily give the following result.

Corollary 3.4. *All solutions of equation* (3.149) *are stable in the mean square sense and with probability* 1. *Moreover, for almost all $\omega \in \Omega$, solution $y = y(t,\omega)$ will tend to zero or to one of the nontrivial periodic solutions $A\cos(at+\varphi)$ of equation* (3.149) *as* $b(t) \equiv 0$.

The result obtained in Theorem 3.10 can easily be transferred to quasilinear systems.

Indeed, together with system (3.113), consider the following system of quasilinear stochastic differential equations:

$$dy = (A + f(t,y)dt + \sigma(t,y)dW(t)\,, \tag{3.150}$$

where $f(t,y), \sigma(t,y)$ are functions that are jointly continuous and Lipschitz continuous in y. Let also, for $f(t,y)$ and $\sigma(t,y)$ there exist nonnegative functions $\beta(t)$ and $\delta(t)$ such that

$$|f(t,y)| \le \beta(t)|y|, \qquad |\sigma(t,y)| \le \delta(t)|y|$$

for all $t \ge t_0, y \in \mathbf{R}^n$.

Theorem 3.12. *Let solutions of system* (3.113) *be bounded on* $[t_0, \infty)$. *If*

$$\int_0^\infty \beta(t)dt \le K_1 < \infty, \qquad \int_0^\infty \delta^2(t)dt \le K_1 < \infty, \tag{3.151}$$

then system (3.150) *is asymptotically mean square equivalent to system* (3.113).

If the second condition in (3.151) *is replaced with the following:*

$$\int_0^\infty t\delta^2(t)dt \le K_1 < \infty\,,$$

then system (3.150) *is asymptotically equivalent to system* (3.113) *with probability* 1.

In the preceding theorems, the unperturbed system had constant coefficients. Let us now consider the case where such a system has variable coefficients.

Consider the following system:

$$dx = A(t)xdt, \tag{3.152}$$

$$dy = (A(t) + B(t))ydt + D(t)ydW(t)\,, \tag{3.153}$$

where $A(t), B(t), D(t)$ are continuous deterministic functions.

Theorem 3.13. *Let* $X(t, \tau)$ *be a matriciant of system* (3.152). *Suppose that there exists a function* $\varphi(t, \tau) \ge 0$ *such that*

1) $\varphi(t, \tau)$ *is monotone decreasing in* t *and monotone increasing in* τ*;*

2) $\varphi(t, t) \le C$ *for arbitrary* $t \ge 0$*;*

3) $\varphi(t, t/2) \to 0$ *as* $t \to \infty$.

If

$$\|X(t, \tau)\| \le \varphi(t, \tau)\,, \tag{3.154}$$

$$\int_0^\infty \|B(t)\|dt < \infty, \qquad \int_0^\infty t\|D(t)\|^2 dt < \infty\,, \tag{3.155}$$

then system (3.153) *is asymptotically equivalent to system* (3.152) *in mean square sense and with probability* 1.

Proof. A solution $y(t)$ of system (3.153), clearly, satisfies the integral equation

$$y(t) = X(t,t_0)y(t_0) + \int_{t_0}^{t} X(t,\tau)B(\tau)y(\tau)d\tau + \int_{t_0}^{t} X(t,\tau)D(\tau)y(\tau)dW(\tau)\,. \tag{3.156}$$

To every solution $y(t)$ of system (3.153) let there correspond a solution $x(t)$ of system (3.152) with the initial condition $x(t_0) = y(t_0)$.

Let us prove that all solutions of system (3.153), with conditions (3.154) and (3.155) satisfied, will be mean square bounded.

Indeed, relation (3.156) yields the following inequalities:

$$\begin{aligned}
\mathbf{E}|y(t)|^2 &\le 3\|X(t,t_0)\|^2\mathbf{E}|y(t_0)|^2 + 3\mathbf{E}\left|\int_{t_0}^{t} X(t,\tau)B(\tau)y(\tau)d\tau\right|^2 \\
&\quad + 3\mathbf{E}\left|\int_{t_0}^{t} X(t,\tau)D(\tau)y(\tau)dW(\tau)\right|^2 \le 3\varphi^2(t,t)\mathbf{E}|y(t_0)|^2 \\
&\quad + 3\mathbf{E}\left(\int_{t_0}^{t} \|X(t,\tau)\|\|B(\tau)\||y(\tau)|d\tau\right)^2 + 3\int_{t_0}^{t} \|X(t,\tau)\|^2\|D(\tau)\|^2\mathbf{E}|y(\tau)|^2 d\tau \\
&\le 3\varphi^2(t,t)\mathbf{E}|y(t_0)|^2 + 3\int_{t_0}^{t} \|X(t,\tau)\|\|B(\tau)\|d\tau \int_{t_0}^{t} \|X(t,\tau)\|\|B(\tau)\|\mathbf{E}|y(\tau)|^2 d\tau \\
&\quad + 3\int_{t_0}^{t} \|X(t,\tau)\|^2\|D(\tau)\|^2\mathbf{E}|y(\tau)|^2 d\tau \le 3\varphi^2(t,t)\mathbf{E}|y(t_0)|^2 \\
&\quad + 3\varphi^2(0,t_0)\int_{t_0}^{t} \|B(\tau)\|d\tau \int_{t_0}^{t} \|B(\tau)\|\mathbf{E}|y(\tau)|^2 d\tau \\
&\quad + 3\varphi^2(t,t)\int_{t_0}^{t} \|D(\tau)\|^2\mathbf{E}|y(\tau)|^2 d\tau\,.
\end{aligned}$$

Now, using conditions (3.154), (3.155), the Gronwall–Bellman inequality we get

$$\mathbf{E}|y(t)|^2 \le 3C^2\mathbf{E}|y(t_0)|^2 e^{3C^2\left(\left(\int_0^\infty \|B(\tau)\|d\tau\right)^2 + \int_0^\infty \|D(\tau)\|^2 d\tau\right)} \le \widetilde{K}\mathbf{E}|y(t_0)|^2.$$

Let us estimate the square of expectation of the difference between the corresponding solutions $x(t)$ and $y(t)$. Since

$$x(t) = X(t, t_0)x(t_0),$$

where $x(t_0) = y(t_0)$, formula (3.156) gives

$$\begin{aligned}
\mathbf{E}|x(t) - y(t)|^2 &= \mathbf{E}\left|\int_{t_0}^{t} X(t,\tau)B(\tau)y(\tau)d\tau + \int_{t_0}^{t} X(t,\tau)D(\tau)y(\tau)dW(\tau)\right|^2 \\
&\le 2\mathbf{E}\left|\int_{t_0}^{t} X(t,\tau)B(\tau)y(\tau)d\tau\right|^2 + 2\mathbf{E}\left|\int_{t_0}^{t} X(t,\tau)D(\tau)y(\tau)dW(\tau)\right|^2 \\
&\le 2\mathbf{E}\left(\int_{t_0}^{t} \|X(t,\tau)\|\|B(\tau)\||y(\tau)|d\tau\right)^2 \\
&\quad + 2\int_{t_0}^{t} \|X(t,\tau)\|^2\|D(\tau)\|^2\mathbf{E}|y(\tau)|^2 d\tau\,.
\end{aligned}$$

Let us estimate each of the two terms in the last inequality. For the first term, we have

$$\begin{aligned}
&2\mathbf{E}\left(\int_{t_0}^{t} \|X(t,\tau)\|\|B(\tau)\||y(\tau)|d\tau\right)^2 \\
&\le 2\widetilde{K}\mathbf{E}|y(t_0)|^2\left(\int_{t_0}^{t} \|X(t,\tau)\|\|B(\tau)\|d\tau\right)^2 \\
&\le 2\widetilde{K}\mathbf{E}|y(t_0)|^2\left(\int_{t_0}^{t} \varphi(t,\tau)\|B(\tau)\|d\tau\right)^2 \\
&= 2\widetilde{K}\mathbf{E}|y(t_0)|^2\left(\int_{t_0}^{t/2} \varphi(t,\tau)\|B(\tau)\|d\tau + \int_{t/2}^{t} \varphi(t,\tau)\|B(\tau)\|d\tau\right)^2
\end{aligned}$$

$$\leq 2\widetilde{K}\mathbf{E}|y(t_0)|^2 \left(\int_{t_0}^{t/2} \varphi(t,t/2)\|B(\tau)\|d\tau + \int_{t/2}^{t} \varphi(t,t)\|B(\tau)\|d\tau \right)^2 \to 0, \quad t \to \infty .$$

A similar reasoning shows that the second term tends to zero. Asymptotic equivalence is proved as in Theorem 3.10. □

Remark 3. If the conditions of the previous theorem are satisfied, it is clear that the unperturbed system (3.152) is asymptotically stable. Theorem 3.13 then asserts that rapidly decaying permanently acting random perturbations do not change the stability property of the system that is asymptotically stable. A similar result using the Lyapunov function method was obtained by Khasmins'kii in [70, p. 308].

The following is an application example for Theorem 3.13.

Example 2. Consider the following system of ordinary differential equations:

$$d\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2t & 0 \\ 0 & -2t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dt\,, \tag{3.157}$$

for which the matriciant has the form

$$X(t,\tau) = \begin{pmatrix} e^{-t^2+\tau^2} & 0 \\ 0 & e^{-t^2+\tau^2} \end{pmatrix}.$$

Consider a perturbed stochastic system,

$$d\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(\begin{pmatrix} -2t & 0 \\ 0 & -2t \end{pmatrix} + B(t) \right) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} dt + D(t) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} dW(t)\,, \tag{3.158}$$

where the matrices $B(t)$ and $D(t)$ are the same as in Example 1. The conditions of Theorem 3.13 for systems (3.157) and (3.158) can be easily checked. Consequently, we see that systems (3.158) and (3.157) are asymptotically equivalent.

3.7 Conditions for asymptotic equivalence of nonlinear systems

We will now turn to a study of asymptotic equivalence of nonlinear systems.

Consider the following differential system:

$$dx = f(t,x)dt\,. \tag{3.159}$$

Together with system (3.159), we will also consider the system of stochastic differential equations,

$$dy = f(t, y)dt + \sigma(t, y)dW(t)\,. \tag{3.160}$$

on a probability space $(\Omega, \mathbf{F}, P)$ with filtration $\{\mathcal{F}_t, t \geq 0\} \subset \mathbf{F}$.

Here $f(t, x), \sigma(t, y) \in C(\mathbf{R}_+ \times \mathbf{R}^\mathbf{n})$ are functions that satisfy the following conditions:

a) there exists a positive constant L such that the estimate

$$|f(t, x) - f(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq L|x - y| \tag{3.161}$$

holds for arbitrary $x, y \in \mathbf{R}^\mathbf{n}$, $t \in [0, \infty)$;

b) there exists a positive constant A such that the inequality

$$|f(t, x)| \leq A(1 + |x|) \tag{3.162}$$

holds for arbitrary $x \in \mathbf{R}^\mathbf{n}$ and $t \in [0, \infty)$;

c) there exists a function $\alpha(t)$, bounded on $t \in [0, \infty)$, such that

$$|\sigma(t, x)| \leq \alpha(t)(1 + |x|) \tag{3.163}$$

for arbitrary $x \in \mathbf{R}^\mathbf{n}$ and arbitrary $t \in [0, \infty)$.

We will find sufficient conditions for the Ito system (3.160) of stochastic differential equations to be equivalent in the mean square sense and with probability 1 to system (3.159) of ordinary differential equations.

Theorem 3.14. *Let solutions $x(t)$ of system* (3.159) *satisfy the following conditions:*

1) *there exists a constant $K_1 \geq 0$ such that*

$$|x(t)| \leq K_1|x(0)| \tag{3.164}$$

for arbitrary $t \geq 0$;

2) *conditions* (3.161) *and* (3.163) *hold and*

$$\alpha(t) \leq K_2 e^{-\gamma t}$$

for $t \geq 0$, where K_2, γ are positive constants that do not depend on t and such that $\gamma > L$.

Then we have the following:

a) *system* (3.160) *is asymptotically mean square equivalent to system* (3.159);

b) *system* (3.160) *is asymptotically equivalent to system* (3.159) *with probability* 1.

Proof. By the conditions of the theorem, a solution $y(t)$ of system (3.160) exists for $t \geq 0$ and is unique. The proof will be split into several steps.

I) Let us find an auxiliary estimate for solutions of system (3.159).

Let $x_1(t)$ and $x_2(t)$ be two such solutions of system (3.159). Then using the Gronwall–Bellman lemma we get the needed estimate,

$$|x_1(t) - x_2(t)| \leq |x_1(s) - x_2(s)| e^{L|t-s|}. \tag{3.165}$$

II) Consider an arbitrary fixed solution $y(t)$ of system (3.160).

Let $\{x_n(t) \mid n \geq 0\}$ be a sequence of solutions of system (3.159) such that

$$x_n(n) = y(n).$$

For $t \in [n, n+1]$, we have the estimates

$$\begin{aligned}
\mathbf{E}|x_n(t) - y(t)|^2 &\leq 3\mathbf{E}\left|\int_n^t \{f(\tau, x_n(\tau)) - f(\tau, y(\tau))\} d\tau\right|^2 \\
&+ 3\mathbf{E}\left|\int_n^t \{\sigma(\tau, x_n(\tau)) - \sigma(\tau, y(\tau))\} dW(\tau)\right|^2 + 3\mathbf{E}\left|\int_n^t \sigma(\tau, x_n(\tau)) dW(\tau)\right|^2 \\
&\leq 6L^2 \int_n^t \mathbf{E}|x_n(\tau) - y(\tau)|^2 d\tau + 3 \int_n^{n+1} \alpha^2(\tau)(2 + 2\mathbf{E}|x_n(\tau)|^2) d\tau \\
&\leq 6L^2 \int_n^t E|x_n(\tau) - y(\tau)|^2 d\tau + 6K_2^2 \int_n^{n+1} e^{-2\gamma\tau}(1 + K_1^2 \mathbf{E}|x_n(0)|^2) d\tau\,.
\end{aligned}$$

Using the Gronwall–Bellman lemma we get from the above that

$$\begin{aligned}
\mathbf{E}|x_n(t) - y(t)|^2 &\leq 6K_2^2 e^{6L^2} \int_n^{n+1} e^{-2\gamma\tau}(1 + K_1^2 \mathbf{E}|x_n(0)|^2) d\tau \\
&\leq 6K_2^2 e^{6L^2}(1 + K_1^2 \mathbf{E}|x_n(0)|^2) e^{-2\gamma n}\,.
\end{aligned} \tag{3.166}$$

By substituting $t = n + 1$ into the latter inequality, we get

$$\begin{aligned}\mathbf{E}|x_n(n+1) - y(n+1)|^2 &= \mathbf{E}|x_n(n+1) - x_{n+1}(n+1)|^2 \\ &\le 6K_2^2 e^{6L^2}(1 + K_1^2 \mathbf{E}|x_n(0)|^2)e^{-2\gamma n}\,.\end{aligned}$$

Using inequality (3.165) we get

$$\begin{aligned}\mathbf{E}|x_n(0) - x_{n+1}(0)|^2 &\le e^{2L(n+1)}\mathbf{E}|x_n(n+1) - x_{n+1}(n+1)|^2 \\ &\le C_1^2(1 + K_1^2\mathbf{E}|x_n(0)|^2)e^{-2(\gamma-L)n}\,, \end{aligned} \tag{3.167}$$

where $C_1^2 := 6K_2^2 e^{6L^2+2L}$.

III) Let us now show that the sequence $\mathbf{E}|x_n(0)|^2$ is bounded. To this end, let us introduce a norm by setting $\|\cdot\| := \sqrt{\mathbf{E}|\cdot|^2}$. Then inequality (3.167) yields

$$\|x_n(0) - x_{n+1}(0)\| \le C_1(1 + K_1\|x_n(0)\|)e^{-(\gamma-L)n}\,. \tag{3.168}$$

Using inequality (3.168) we get

$$\begin{aligned}1 + K_1\|x_n(0)\| &= 1 + K_1\|x_n(0) - x_{n-1}(0) + x_{n-1}(0)\| \\ &\le 1 + K_1\|x_{n-1}(0)\| + K_1\|x_n(0) - x_{n-1}(0)\| \\ &\le 1 + K_1\|x_{n-1}(0)\| + C_1K_1(1 + K_1\|x_{n-1}(0)\|)e^{-(\gamma-L)(n-1)} \\ &= (1 + K_1\|x_{n-1}(0)\|)(1 + C_1K_1e^{-(\gamma-L)(n-1)}) \le \cdots \\ &\le (1 + K_1\|x_0(0)\|)\prod_{k=1}^{n}(1 + C_1K_1e^{-(\gamma-L)(n-k)})\,.\end{aligned}$$

Since

$$\ln\prod_{k=1}^{n}(1 + C_1K_1e^{-(\gamma-L)(n-k)}) \le \frac{C_1K_1e^{\gamma-L}}{e^{\gamma-L}-1}(1 - e^{-n(\gamma-L)})\,,$$

we have that

$$1 + K_1\|x_n(0)\| \le C_2(1 + K_1\|x_0(0)\|)\,, \tag{3.169}$$

where

$$C_2 = \exp\frac{C_1K_1e^{\gamma-L}}{e^{\gamma-L}-1}.$$

So, for $t \in [n, n+1]$, using estimates (3.166), and (3.169) we have

$$||x_n(t) - y(t)|| \le C_3(1 + K_1\|x_0(0)\|)e^{-\gamma n}, \tag{3.170}$$

where $C_3 := C_1C_2e^{-L}$.

IV) Let us now prove part a) of Theorem 3.14. It follows from inequality (3.168) that

$$\|x_n(0) - x_{n+1}(0)\| \leq C_1 C_2 (1 + K_1 \|x_0(0)\|) e^{-(\gamma - L)n}.$$

So the limit $x_\infty = \lim_{n\to\infty} x_n(0)$ exists in the mean square sense.

Define a solution $x_\infty(t)$ of system (3.159) as a solution of the Cauchy problem with the initial condition $x_\infty(0) = x_\infty$.

Let arbitrary $t \geq 0$ and $n \in \mathbf{N}$ be such that $n \leq t \leq n+1$. Then

$$\mathbf{E}|y(t) - x_\infty(t)|^2 \leq 2\mathbf{E}|y(t) - x_n(t)|^2 + 2\mathbf{E}|x_n(t) - x_\infty(t)|^2.$$

Inequality (3.170) shows that the expression $\mathbf{E}|y(t) - x_n(t)|^2$ tends to zero as $n \to \infty$.

Let $T > 0$ be arbitrary. Since a solution of system (3.159) is continuous with respect to the initial condition, it follows that $x_n(t)$ converges uniformly on the segment $[0, T]$ to $x_\infty(t)$ as $n \to \infty$ in the mean square sense.

Then we have

$$\begin{aligned}
&\|x_n(t) - x_\infty(t)\| \leq \exp L \|x_n(n) - x_\infty(n)\| \\
&\quad \leq \|\sum_{k=n}^{\infty} (x_{k+1}(n) - x_k(n))\| \exp L \\
&\quad \leq \exp L \sum_{k=n}^{\infty} \exp L(k-n) \|x_{k+1}(k) - x_k(k)\| \\
&\quad \leq \exp L \sum_{k=n}^{\infty} \|x_{k+1}(n) - x_k(n)\| \leq C_3 (1 + K_1 \|x_0(0)\|) \sum_{k=n}^{\infty} e^{-\gamma k} e^{L(k-n)} \\
&\quad = C_3 (1 + K_1 \|x_0(0)\|) e^{-\gamma n} \sum_{k=0}^{\infty} e^{-(\gamma - L)k},
\end{aligned}$$

so $\|x_n(t) - x_\infty(t)\| \to 0$ as $n \to \infty$. Hence, we finally get that

$$\mathbf{E}|y(t) - x_\infty(t)|^2 \to 0, \qquad t \to \infty,$$

which proves claim a) of Theorem 3.14.

V) Let us now consider part b) of the theorem. For the sequence $\{x_n(t) | n \geq 0\}$, $t \in [n, n+1]$, introduced above and a sequence of positive real numbers, $\{\varepsilon_n | n \geq 0\}$, let us estimate the expression

$$\mathbf{P}\left\{ \sup_{t \in [n, n+1]} |x_n(t) - y(t)| \geq \varepsilon_n \right\}.$$

We have

$$\mathbf{P}\left\{\sup_{t\in[n,n+1]}|x_n(t)-y(t)|\geq\varepsilon_n\right\}$$

$$=\mathbf{P}\left\{\sup_{t\in[n,n+1]}\left|\int_n^t f(\tau,x_n(\tau))d\tau-\int_n^t f(\tau,y(\tau))d\tau-\int_n^t\sigma(\tau,y(\tau))dW(\tau)\right|\geq\varepsilon_n\right\}$$

$$\leq\mathbf{P}\left\{\sup_{t\in[n,n+1]}\left|\int_n^t(f(\tau,x_n(\tau))-f(\tau,y(\tau)))d\tau\right|\geq\frac{\varepsilon_n}{3}\right\}$$

$$+\mathbf{P}\left\{\sup_{t\in[n,n+1]}\left|\int_n^t(\sigma(\tau,x_n(\tau))-\sigma(\tau,y(\tau)))dW(\tau)\right|\geq\frac{\varepsilon_n}{3}\right\}$$

$$+\mathbf{P}\left\{\sup_{t\in[n,n+1]}\left|\int_n^t\sigma(\tau,x_n(\tau))dW(\tau)\right|\geq\frac{\varepsilon_n}{3}\right\}. \tag{3.171}$$

Let us estimate each term in the right-hand side of the last inequality. Using the Chebyshev inequality, estimates (3.165) and (3.170) we get

$$\mathbf{P}\left\{\sup_{t\in[n,n+1]}\left|\int_n^t(f(\tau,x_n(\tau))-f(\tau,y(\tau)))d\tau\right|\geq\frac{\varepsilon_n}{3}\right\}$$

$$\leq\frac{3}{\varepsilon_n}\mathbf{E}\sup_{t\in[n,n+1]}\left|\int_n^t(f(\tau,x_n(\tau))-f(\tau,y(\tau)))d\tau\right|$$

$$\leq\frac{3L}{\varepsilon_n}\int_n^{n+1}\mathbf{E}|x_n(\tau)-y(\tau)|d\tau\leq\frac{3L}{\varepsilon_n}C_3(1+K_1\|x_0(0)\|)e^{-\gamma n}.$$

Using the properties of a stochastic integral and inequalities (3.165) and (3.170) we get the following for the second term in (3.171):

$$\mathbf{P}\left\{\sup_{t\in[n,n+1]}\left|\int_n^t(\sigma(\tau,x_n(\tau))-\sigma(\tau,y(\tau)))dW(\tau)\right|\geq\frac{\varepsilon_n}{3}\right\}$$

$$\leq\frac{9}{\varepsilon_n^2}\int_n^{n+1}\mathbf{E}|\sigma(\tau,x_n(\tau))-\sigma(\tau,y(\tau))|^2d\tau\leq\frac{9L^2}{\varepsilon_n^2}\int_n^{n+1}\mathbf{E}|x_n(\tau)-y(\tau)|^2d\tau$$

$$\leq \frac{9L^2}{\varepsilon_n^2} C_3^2 (1 + K_1 \|x_0(0)\|)^2 e^{-2\gamma n}.$$

Using inequality (3.169), let us estimate the third term in (3.171). We have

$$\mathbf{P}\left\{ \sup_{t\in[n,n+1]} \left| \int_n^t \sigma(\tau, x_n(\tau)) dW(\tau) \right| \geq \frac{\varepsilon_n}{3} \right\} \leq \frac{9}{\varepsilon_n^2} \int_n^{n+1} \mathbf{E}|\sigma(\tau, x_n(\tau))|^2 d\tau$$

$$\leq \frac{2\cdot 9}{\varepsilon_n^2} \int_n^{n+1} \alpha^2(\tau)(1 + \mathbf{E}|x_n(\tau)|^2) d\tau \leq \frac{18K_2^2}{\varepsilon_n^2} \int_n^{n+1} e^{-2\gamma\tau}(1 + K_1^2 \mathbf{E}|x_n(0)|^2) d\tau$$

$$\leq \frac{2\cdot 18K_2^2}{\varepsilon_n^2} C_2^2 (1 + K_1^2 \mathbf{E}|x_0(0)|^2) e^{-2\gamma n}.$$

Choose a sequence $\varepsilon_n := e^{-\frac{\gamma+L}{2}n}$. Then it is easily seen that

$$\mathbf{P}\left\{ \sup_{t\in[n,n+1]} |x_n(t) - y(t)| \geq \varepsilon_n \right\} \leq Ae^{-\frac{\gamma-L}{2}n} + Be^{-(\gamma-L)n},$$

where A and B are constants independent of n.

It is clear that a series with the members $Ae^{-\frac{\gamma-L}{2}n} + Be^{-(\gamma-L)n}$ is convergent and, hence, it follows from the Borel–Cantelli lemma that there is a positive entire random variable $N = N(\omega)$ such that, for arbitrary $n \geq N(\omega)$,

$$\sup_{t\in[n,n+1]} |x_n(t) - y(t)| \leq e^{-\frac{\gamma+L}{2}n}$$

for almost all $\omega \in \Omega$. For $t = n+1$, this gives that

$$|x_n(n+1) - x_{n+1}(n+1)| \leq e^{-\frac{\gamma+L}{2}n}$$

with probability 1. The latter inequality, together with inequality (3.165), give

$$|x_n(0) - x_{n+1}(0)| \leq e^L e^{-\frac{\gamma-L}{2}n},$$

which implies existence of the limit $x_\infty = \lim_{n\to\infty} x_n(0)$ with probability 1.

The rest of the proof of part b) of this theorem is carried out similarly replacing the mean square convergence with convergence with probability 1. This leads to the finial relation

$$\mathbf{P}\{\lim_{t\to\infty} |x_\infty(t) - y(t)| = 0\} = 1,$$

which finishes the proof. $\square$

3.8 Comments and References

Section 3.1. One of central problems of the theory of linear equations is the question of exponential dichotomy of solutions, since this includes a study of both stability of solutions and conditions for their unbounded growth. A study of dichotomy properties has originated in the works of J. Hadamard and O. Perron. Nowadays there is a large number of works dealing with this question for linear and even nonlinear systems, see e.g. Arnold [11], Bogolyubov, Mitropolsky, Samoilenko [22], Samoilenko [140, 141], Bronshtein [27], Golets', Kulik [56], Daletsky, Krein [36], Massera, Sheffer [100], Mitropolsky, Samoilenko, Kulik [106], and other. It turned out that dichotomy properties are closely related to the presence of bounded solutions of a corresponding nonhomogeneous system. A study of bounded solutions using Lyapunov functions was conducted in the works of Maizel [95] and Malkin [98].

Let us also mention interesting works of Palmer [119] and Boichuk [23], where the dichotomy properties were linked to the Noetherian property of an operator specially constructed from the linear system.

The main questions dealt with in the theory of ordinary differential equations are fully topical for stochastic linear systems, although they are much more complicated. Even for systems with constant coefficients, as opposed to the deterministic case, it is already impossible to explicitly write their solutions. However, as it has been mentioned by many authors, see e.g. Leibowits [92], Gikhman, Skorokhod [51] and others, the problem of finding moments of orders 1, 2, 3, ... can be reduced to solving an auxiliary deterministic system of linear differential equations. Here, the equation for the first moments has the same dimension as the original one. However, the particularity of the probability case fully shows for the system of equations for the second moments. For example, it is not hard to give examples where the first moments tend to zero at infinity while the second moments are unbounded. For this reason, it is important in the linear theory to study the behavior of, particularly, the second moments or, in other words, to make the correlation analysis of the linear stochastic Ito systems. A fairly complete exposition of such an analysis can be found in the monograph of Pugachev [129].

It should be noted that a number of methods used in the above cite works are not very effective from a practical points of view because of the large dimension of the auxiliary deterministic systems. In the works of Khaminsky [70], Tsar'kov [186], the mean square stability of solutions of linear stochastic systems is studied in terms of the stochastic systems themselves by using the Lyapunov function method. Let us also mention the works of Arnold [9], Arnold, Oeljeklaus and Pardoux [10] where the authors constructed a theory of stochastic Lyapunov exponents.

As opposed to the theory of ordinary differential equations, questions of the mean square exponential dichotomy have not been sufficiently studied for stochastic systems. The authors know few related results. Let us mention the monograph of Tsar'kov [186, p. 296], where conditions for mean square exponential dichotomy were obtained for stochastic systems and systems with delay in the case where the matrix of the system is either constant or periodic see also Tappe [182].

The results given in this section were obtained by Stanzhytskij in [175].

Section 3.2. Exponential dichotomy was studied in terms of Lyapunov functions for ordinary differential equations, for example, in the works of Golets, Kulik [56], Maizel [95], Samoilenko [140], in the monographs of Daletsky, Krein [36], Masser, Sheffer [100], Mitropolsky, Samoilenko, Kulik [106]. There are few results in this direction for stochastic systems. In the work of Stanzhytskij [168], which is exposed in this section, the author studies dichotomy of linear stochastic Ito systems by using the method of quadratic forms applied directly to the system under consideration. In the work of Stanzhytskij, Krenevich [174], the mean square dichotomy was studied for a linear system with random initial conditions. Also, in this work, it was proved that for a linear dichotomous stochastic system there exists a quadratic form such that its generating operator along the system is a negative definite quadratic form, that is, there were obtained necessary and sufficient conditions for a system to be dichotomous.

Sections 3.3—3.4. The mean square exponential dichotomy is closely related to existence of mean square bounded solutions of the corresponding nonhomogeneous system. The questions of existence of bounded solutions were treated in works of many authors. The first results were obtained by T. Morozan in [113, 111, 112]. Stationary solutions of linear stochastic systems with constant coefficients were studied in Arató [7]. Similar results for infinite dimensional case were obtained by Dorogovtsev in [41]. However, there are still some open questions related to the existence and the method of finding bounded solutions. Especially, this is the case for linear systems with variable matrix and for weakly nonlinear systems. The content of these sections is exposed in the work of Stanzhytskij [171].

Section 3.5. As we have already mentioned, the mean square dichotomy is closely related to mean square bounded solutions of the nonhomogeneous system. Existence of solutions, bounded on a semiaxis, of the nonhomogeneous system for an arbitrary mean square bounded nonlinearity implies exponential mean square dichotomy on the semiaxis. Getting the converse result, as opposed to the deterministic case, one faces fundamental difficulties related to the definition of a solution of the stochastic Ito equation, where we need it to agree with the corresponding flow of σ-algebras. However, if one somewhat generalizes the notion of the solution, then such a result can be obtained, which is done in the work of Il'chenko [61]. The section gives an exposition of this result.

Sections 3.6—3.7. For ordinary differential equation, the Levinson theorem, see e.g. Demidovich [38], is a classical result that gives conditions for asymptotic equivalence of linear systems. The authors are unaware of results of such type for stochastic equations, other than the ones given in these section. Results on comparison of the behavior of solutions of stochastic systems and those of deterministic systems are contained, for example, in the works of Buldygin, Koval' [31], Buldygin, Klesov, Steinebach [30], Kulinich [87]. The content of these sections is based on the works of Kulinich [83, 84].

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Chapter 4

Extensions of Ito systems on a torus

In this chapter, we will study invariant tori that constitute supports for nonlinear oscillations, and the related linear and nonlinear stochastic extensions of dynamical systems on a torus.

In Section 4.1, we use local coordinates on a neighborhood of the torus to write the considered stochastic system in a special form in terms of the amplitude and the phase coordinates. This permits us to use Lyapunov functions, obtain conditions for probability stability of the torus of the initial stochastic system.

Section 4.2 deals with random invariant tori for linear stochastic extensions of dynamical systems on a torus. We introduce a notion of a stochastic Green function for the problem on the invariant tori, which allows us to obtain an integral representation for the random invariant torus by using the stochastic Ito integral.

In Section 4.3, using an integral representation for the torus we obtain conditions for smoothness of the torus depending on the initial conditions, and write the mean square derivative. As a generalization, we obtain conditions for existence of the mean square derivatives of higher orders.

Section 4.4 considers nonlinear stochastic extensions of dynamical systems on a torus. Assuming the nonlinearity in a neighborhood of the torus to be small, we obtain existence conditions for random invariant tori for nonlinear systems. The corresponding result is proved by linearizing the initial system and constructing an iteration procedure.

Section 4.5 gives an ergodic type theorem on the behavior of the trajectories of the stochastic system in a neighborhood of the invariant torus.

4.1 Stability of invariant tori

Consider a system of stochastic Ito equations,

$$dx = X(x)dt + \ Y(x)dW(t)\,, \tag{4.1}$$

where $x \in \mathbf{R}^n$, $t \geq 0$, X is a vector in $\mathbf{R}^n$, Y an $(n \times m)$-dimensional measurable matrix, $W(t)$ an m-dimensional Wiener process with independent increments. We assume that the coefficients of the system satisfy conditions for existence and strong uniqueness of a solution of the Cauchy problem for $x_0 \in \mathbf{R}^n$, $t \geq 0$, see e.g. [54, pp. 234, 236].

Let N be a toroidal manifold defined by the equation $x = f(\varphi)$, $\varphi \in \Im_m$, and rank $\{\frac{\partial f(\varphi)}{\partial \varphi}\} = m$. Also assume that the matrix $\frac{\partial f(\varphi)}{\partial \varphi}$ can be augmented to define a periodic basis in $\mathbf{R}^n$, and $B(\varphi)$ is the augmentation matrix, see [139, p. 38].

It is convenient, when studying the behaviour of solutions of system (4.1) in a neighborhood of the torus N, as in the deterministic case, to pass from the Cartesian coordinates to local coordinates (φ, h) [139, p. 96] by the formula

$$x = f(\varphi) + B(\varphi)h\,. \tag{4.2}$$

With such a change of coordinates, the torus $x = f(\varphi)$ becomes the torus $h = 0$, which helps to check its invariance, stability, etc.

If f and B are sufficiently smooth, equation (4.1) can be written in a neighborhood of the set N in terms of local coordinates in a way that $|h| < \delta$, $\varphi \in \Im_m$, where δ is a positive number. Such a change of coordinates was used in [173] in the case where $n = 2$. In the general case, the change of coordinates is similar although more complex.

We will assume that system (4.1) has already been written in the local coordinates (φ, h) in some neighborhood of the manifold N, hence it has the form

$$d\varphi = a_1(\varphi, h)dt + b_1(\varphi, h)dW(t), \;\; dh = a_2(\varphi, h)dt + b_2(\varphi, h)dW(t)\,, \tag{4.3}$$

where $t \geq 0$, $\varphi \in \Im_m$, $|h| < \delta$.

In this section, we will study stability in probability of the invariant torus N of system (4.1) or, which is the same thing, stability of the torus $h = 0$, $\varphi \in \Im_m$.

Recall that the set $h = 0$ invariant for a process (φ_t, h_t) is stable in probability if

$$\lim_{|h|\to 0} \mathbf{P}_h \left\{ \sup_{t>0} |h_t| > \varepsilon \right\} = 0\,,$$

where $h_0 = h$ and $\varepsilon > 0$ is arbitrary.

Let functions $A_1(\varphi, h)$, $B_1(\varphi, h)$, and $A_2(\varphi, h)$, $B_2(\varphi, h)$ be defined for $\varphi \in \Im_m, h \in R^{n-m}$ and such that the system

$$\begin{aligned} d\varphi &= A_1(\varphi, h)dt + B_1(\varphi, h)dW(t)\,, \\ dh &= A_2(\varphi, h)dt + B_2(\varphi, h)dW(t) \end{aligned} \tag{4.4}$$

satisfies a condition for regularity of solution of the Cauchy problem for arbitrary $\varphi_0 \in \Im_m$, $h_0 \in R^{n-m}$. Here $a_1(\varphi, h) = A_1(\varphi, h)$, $b_1(\varphi, h) = B_1(\varphi, h)$, $a_2(\varphi, h) = A_2(\varphi, h)$, $b_2(\varphi, h) = B_2(\varphi, h)$ for $\varphi \in \Im_m$, $|h| \le \delta_0$, and some $\delta_0 < \delta$.

Let us now remark that systems (4.1) and (4.4) are equivalent for $|h| \le \delta_0$ in the sense that the solution $x_t(x_0)$ of system (4.1), $x_0(x_0) = x_0$, trajectory-wise coincides with a solution of system (4.4) until $\tau_{(\varphi_0, h_0)}$, the time the solution crosses the boundary of the region $|h| < \delta_0$ for the first time. Here (φ_0, h_0) are local coordinates corresponding to x_0. Hence, the trajectories of the random process (φ_t, h_t) such that $|h| < \delta_0$ coincide with the trajectories $x_t(x_0)$ for all $t \ge 0$.

Using the above and estimate (7.11) in [139, p.98] we see that stability in probability of the torus N is equivalent to stability in probability of the set $h = 0$ for system (4.4).

Note that, if the set N is invariant for system (4.1), then the corresponding set $h = 0$ is invariant for system (4.4), which is possible only if

$$A_2(\varphi, 0) = 0 \;\; B_2(\varphi, 0) = 0$$

for arbitrary $\varphi \in \Im_m$.

We will assume that N is an invariant set for system (4.1). Let us separate, in (4.4), the terms linear in h in a neighborhood of $h = 0$,

$$\begin{aligned} A_1(\varphi, h) &= A_1(\varphi, 0) + \frac{\partial A_1(\varphi, 0)}{\partial h} h + \dots\,, \\ A_2(\varphi, h) &= \frac{\partial A_2(\varphi, 0)}{\partial h} h + \dots\,, \\ B_1(\varphi, h) &= B_1(\varphi, 0) + \frac{\partial B_1(\varphi, 0)}{\partial h} h + \dots\,, \\ B_2(\varphi, h) &= \frac{\partial B_2(\varphi, 0)}{\partial h} h + \dots\,. \end{aligned}$$

We write the system

$$d\varphi = a_0(\varphi)dt + Q_{01}(\varphi)dW(t), \quad dh = P_0(\varphi)hdt + Q_{02}(\varphi)hdW(t) \tag{4.5}$$

denoting

$$a_0(\varphi) = A_1(\varphi, 0), \qquad Q_{01}(\varphi) = B_1(\varphi, 0),$$

$$P_0(\varphi) = \frac{\partial A_2(\varphi, 0)}{\partial h}, \qquad Q_{02}(\varphi) = \frac{\partial B_2(\varphi, 0)}{\partial h},$$

and call it, as in [139, p. 99], a *system in variations* for the invariant torus $h = 0$ for system (4.3).

It is clear that equations in variations are defined if the functions $X(x), Y(x)$ have continuous partial derivatives in x in a neighborhood of the manifold N and $f(\varphi)$, $B(\varphi)$ are twice continuously differentiable with respect to φ.

Theorem 4.1. *Suppose that system* (4.1) *and the torus N satisfy the above conditions on smoothness, invariance, and possibility to introduce the local coordinates. Assume that in a neighborhood of the torus $h = 0$ there exists a positive definite quadratic form $V = (S(\varphi)h, h)$ with positive definite symmetric matrix $S(\varphi) \in C^2(\Im_m)$ such that the quadratic form L_0V satisfies the estimate*

$$L_0V \le -\beta|h|^2$$

for all $\varphi \in \Im_m$, $|h| < \delta_0$, where L_0 is a generating operator for the Markov process in (4.5).

Then the manifold N is stable in probability in the sense that for arbitrary ε_1, ε_2 there exists $r > 0$ such that

$$\mathbf{P}\left\{\sup_{t\ge 0} \rho(x_t, N) > \varepsilon_1\right\} < \varepsilon_2 \tag{4.6}$$

for $\rho(x_0, N) < r$.

Proof. It follows from [173] that the torus $h = 0$, $\varphi \in \Im_m$ for system (4.4) is stable in probability. This means that for arbitrary $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exists $r > 0$ such that

$$\mathbf{P}\left\{\sup_{t\ge 0} |h_t| > \varepsilon_1\right\} < \varepsilon_2 \tag{4.7}$$

for $|h_0| < r$. By taking $\varepsilon_1 < \delta_0$, $r < \delta_0$, we can get that

$$\mathbf{P}\left\{\omega \in \Omega : \sup_{t\ge 0} |h_t(\omega)| \le \varepsilon_1\right\} > 1 - \varepsilon_2 \,. \tag{4.8}$$

Hence, the trajectories of the process $(\varphi_t,\ h_t)$ do not leave the ε_1-neighborhood of the set $h = 0$ for these ω, that is, they do not reach the boundary of the domain $|h| < \delta_0$. This, by the above, implies that these trajectories coincide with trajectories of the corresponding process x_t defined by system (4.1) for all $t \geq 0$. Using (4.8) we see that estimate (4.6) holds. $\square$

Remark. As in [139, p. 98], Theorem 4.1 will be called the stability theorem in the first order approximation for invariant tori for stochastic systems.

Let us now prove an auxiliary result.

Lemma 4.1. *If there exists a positive definite quadratic form* $V = (S(\varphi)h, h)$, $S(\varphi) \in C^2(\Im_m)$, *for system* (4.4) *such that*

$$LV \leq -\beta|h|^2$$

for some $\beta > 0$ *and all* $\varphi \in \Im_m$, $h \in R^{n-m}$, *then the torus* $h = 0$ *for system* (4.4) *is mean square exponentially stable and the following estimate holds with probability* 1:

$$|h_t(\varphi_0, h_0)| \leq K(h_\tau)\exp\{-\gamma(t-\tau)\}\,, \tag{4.9}$$

where the random variable $K(h_\tau)$ *is finite with probability* 1.

Proof. The proof of this lemma is conducted as the proof of the result in [70, p. 232]. Since the quadratic form $(S(\varphi)h, h)$ is positive definite and the matrix $S(\varphi)$ is periodic, there exist $\gamma_0 > 0$, $\gamma^0 > 0$ such that

$$\gamma_0|h|^2 \leq\ (S(\varphi)h, h) \leq \gamma^0|h|^2 \tag{4.10}$$

for $\varphi \in \Im_m$, $h \in R^{n-m}$.

Note that the conditions of the lemma imply that solutions of equation (4.4) are regular for $t \geq 0$. The existence and uniqueness theorem for (4.4) gives that $M(S(\varphi_t)h_t, h_t)$ exists for $t \geq 0$.

By applying the integral form of the Ito formula to $(S(\varphi_t)h_t, h_t)$ and calculating expectation, we get

$$\mathbf{E}(S(\varphi_t)h_t, h_t) - \ \mathbf{E}(S(\varphi_\tau)h_\tau, h_\tau) = \int_\tau^t \mathbf{E}L(S(\varphi_u)h_u, h_u)du\,.$$

Differentiating it with respect to t, for $t \geq \tau$, we get

$$\frac{d}{dt}\mathbf{E}(S(\varphi_t)h_t, h_t) = \mathbf{E}L(S(\varphi_t)h_t, h_t)\,.$$

Using 4.10 we get the inequality

$$\frac{d}{dt}\mathbf{E}(S(\varphi_t)h_t, h_t) \leq -\beta\mathbf{E}|h_t|^2 \leq -\frac{\beta}{\gamma^0}\mathbf{E}(S(\varphi_t)h_t, h_t)\,.$$

Multiplying the left- and the right-hand sides by $\exp\{\frac{\beta}{\gamma^0}(t-\tau)\}$ yields the inequality

$$\frac{d}{dt}\left[\exp\left\{\frac{\beta}{\gamma^0}(t-\tau)\right\}\mathbf{E}(S(\varphi_t)h_t, h_t)\right] \leq 0\,,$$

which, when integrated over $[\tau,\ t]$, gives

$$\mathbf{E}(S(\varphi_t)h_t, h_t) \leq \mathbf{E}(S(\varphi_\tau)h_\tau, h_\tau)\exp\left\{-\frac{\beta}{\gamma^0}(t-\tau)\right\}.$$

Now, using (4.10) we get that

$$\gamma_0\mathbf{E}|h_t|^2 \leq \ \gamma^0\exp\left\{-\frac{\beta}{\gamma^0}(t-\tau)\right\}\mathbf{E}|h_\tau|^2\,, \tag{4.11}$$

and this inequality implies that the torus $h=0$ for system (4.4) is mean square exponentially stable.

To prove estimate (4.9), apply the generating operator L of system (4.4) to the function

$$V_1 = V(\varphi, h)\exp\left\{\frac{\beta}{\gamma^0}(t-\tau)\right\}.$$

Using (4.10) we get

$$\begin{aligned}
LV_1 &= \frac{\beta}{\gamma^0}\exp\left\{\frac{\beta}{\gamma^0}(t-\tau)\right\}V(\varphi,h) + \exp\left\{\frac{\beta}{\gamma^0}(t-\tau)\right\}LV(\varphi,h)\\
&\leq \frac{\beta}{\gamma^0}\exp\left\{\frac{\beta}{\gamma^0}(t-\tau)\right\}(S(\varphi)h,h) - \beta|h|^2\exp\left\{\frac{\beta}{\gamma^0}(t-\tau)\right\}\\
&\leq \frac{\beta}{\gamma^0}\exp\left\{\frac{\beta}{\gamma^0}(t-\tau)\right\}(S(\varphi)h,h) - \frac{\beta}{\gamma^0}\exp\left\{\frac{\beta}{\gamma^0}(t-\tau)\right\}(S(\varphi)h,h) = 0
\end{aligned} \tag{4.12}$$

for all $t > \tau$.

It follows from (4.12), with a use of Lemma 3.1 in [70, p. 110], that the random process $V(\varphi_t, h_t)\exp\{\frac{\beta}{\gamma^0}(t-\tau)\}$ is a supermartingale for $t \geq 0$ with respect to the σ-algebra F_t that enters the definition of a solution of equation (4.1).

As in [70, p. 205], we get that the set $h = 0$, $\varphi \in \Im_m$ is unreachable for the process (φ_t, h_t) if h_0 is nonzero, that is, h_t can be zero for some $t \geq 0$ only with zero probability.

Take arbitrary $t > \tau \geq 0$. It follows from the uniqueness theorem that the solution $(\varphi_t, h_t(h_\tau))$ of system (4.4), for $t \geq \tau$, coincides with the solution (φ_t, h_t). In the same way, one proves that the process $V(\varphi_t, h_t(h_\tau))$ $\exp\{\frac{\beta}{\gamma^0}(t-\tau)\}$ is a supermartingale with respect to the minimal σ-algebra that contains the events $h_\tau, W(s) - W(\tau)$ for $\tau \leq s \leq t$. Since $h_t(h_\tau)$ coincides with h_t for $t \geq \tau$, we have that $V(\varphi_t, h_t(h_\tau)) \exp\{\frac{\beta}{\gamma^0}(t-\tau)\}$ is a positive supermartingale and, hence, as follows from [42], it has a finite limit with probability 1 as $t \to \infty$. This means that the following estimate holds with probability 1 for arbitrary $t \geq \tau$:

$$V(\varphi_t, h_t(h_\tau)) \exp\left\{\frac{\beta}{\gamma^0}(t-\tau)\right\} \leq K(h_\tau)\,,$$

which, with a use of (4.10), leads to inequality (4.9) $\square$

Let us remark that the trajectories of the solution x_t of system (4.1) that do not hit the boundary of the domain $|x - f(\varphi)| < \delta_0$ coincide with the corresponding trajectories (φ_t, h_t) of system (4.4) for $t \geq 0$. This means that, by Lemma 4.1, they satisfy the estimate

$$|x_t(x_t(x_\tau) - f(\varphi)| \leq K(x_\tau) \exp\left\{-\frac{\beta}{\gamma^0}(t-\tau)\right\} \tag{4.13}$$

for arbitrary $t \geq \tau \geq 0$.

Then Theorem 4.1 shows that the probability that this estimate holds can be made arbitrarily close to 1 by choosing the difference $|x_0 - f(\varphi)|$ to be sufficiently small.

The above reasoning can be summed up as the following theorem.

Theorem 4.2. *If the conditions of Lemma 4.1 are satisfied, the invariant set N for system* (4.1) *is stable in probability and for arbitrary $\varepsilon > 0$ there exists $r_0 > 0$ such that, if $\rho(x_0, N) < r_0$, then*

$$\mathbf{P}\{\ |x_t(x_\tau) - f(\varphi)| \leq K(x_\tau) \exp\{-\gamma(t-\tau)\}\} \geq 1 - \varepsilon$$

for arbitrary $t \geq \tau \geq 0$ and some $\gamma > 0$, where the random variable $K(x(\tau))$ is finite almost certainly.

4.2 Random invariant tori for linear extensions

Consider a stochastic linear extension of a dynamical system on a torus $x = f(\varphi),\ \varphi \in \Im_m$, with $\operatorname{rank}\{\frac{\partial f}{\partial \varphi}\} = m$ for arbitrary $\varphi \in \Im_m$, such that the extension is a system of stochastic Ito equations of the form

$$d\varphi = a(\varphi)dt,\ dh = (P(\varphi)h + f(\varphi))dt + g(\varphi)dW(t)\,, \tag{4.14}$$

where $a,\ P,\ f,\ g$ are functions continuous, periodic in φ_i with period 2π, and $\varphi = (\varphi_1, \dots \varphi_m),\ h = (h_1, \dots h_n), \varphi \in \Im_m,\ h \in \mathbf{R}^n,\ W(t)$ is a Wiener process on $\mathbf{R}$ defined on a complete probability space (Ω, F, P).

We will assume that $a(\varphi)$ is Lipschitz continuous, so that the first equation in (4.14) always has a unique solution $\varphi_t(\varphi),\ \varphi_0(\varphi) = \varphi$. By substituting this solution into the second equation in (4.14), we get a system of linear differential Ito equations for h_t,

$$dh_t = (P(\varphi_t(\varphi))h_t + f(\varphi_t(\varphi)))dt + g(\varphi_t(\varphi))dW(t)\,. \tag{4.15}$$

For each $t \in \mathbf{R}$, define the σ-algebra F_t to be the minimal σ-algebra generated by sets of the form

$$\{W(s_2) - W(s_1)\ : s_1 \le s_2 \le t\}\,.$$

Definition 4.1. We say that a random process h_t is a solution of system (4.15) on $\mathbf{R}$ if the following holds.

1) for arbitrary $t \in \mathbf{R}$, the process h_t is F_t-measurable;

2) h_t is continuous on $\mathbf{R}$ with probability 1;

3) for arbitrary $-\infty < t_0 < t_1 < \infty$,

$$\sup_{t \in [t_0,\ t_1]} \mathbf{E}|h_t|^2 < \infty\,;$$

4) for arbitrary $-\infty < t_0 < t_1 < \infty$,

$$h_t = h_{t_0} + \int_{t_0}^{t} (P(\varphi_s(\varphi))h_s + f(\varphi_s(\varphi)))\,ds + \int_{t_0}^{t} g(\varphi_s(\varphi))\,dW(s)$$

with probability 1.

If all the conditions 1)–3) are satisfied for $t \geq 0$, then the random process h_t will be called a *solution* of the Cauchy problem for $t \geq 0$ with an initial F_0-measurable condition h_0 that has finite second moment.

As follows from, e.g., [54, p. 141], if the right-hand sides of system (4.14) satisfy the above conditions, then such a solution exists and is strongly unique.

The first equation in (4.14) can now be interpreted as a dynamical system on the m-dimensional torus $\Im_m$. It describes a certain oscillation process with constant oscillation amplitude, for example, the oscillation of a mathematical pendulum without friction and with no external perturbations. From a physical point of view, system (4.14) describes then a process with the amplitude being perturbed with "white noise" type random perturbations. It is natural to ask what are the conditions so that the system will have solutions that are oscillating in a certain sense ? The answer to this question, as in the deterministic case, will be connected to the existence of invariant tori for the system under consideration. It is clear that existence of deterministic invariant tori $h = u(\varphi)$ for system (4.14) imposes fairly strict conditions on the right-hand sides. In particular, a necessary condition for the torus $h = u(\varphi)$ to be invariant is that $g(\varphi_t) \equiv 0$ on all solutions φ_t of the first equation in (4.14), which means that the influence of the "white noise" on the torus is disregarded. It is thus natural to assume that for systems (4.14) under consideration there exist random invariant tori rather than deterministic ones.

Definition 4.2. A random function $h = u(t, \varphi, \omega)$, $\varphi \in \Im_m,\ \omega \in \Omega$, is called a *random invariant torus* for system (4.14) if

1) $u(t, \varphi + 2\pi k, \omega) = u(t, \varphi, \omega)$ with probability 1 for arbitrary integer vector $k = (k_1, \dots k_m)$;

2) the pair $(\varphi_t(\varphi)$, $h_t = u(t, \varphi_t(\varphi), \omega))$, where $\varphi_t(\varphi)$ is a solution of the first equation in (4.14), is called a *solution* of system (4.14) on $\mathbf{R}$.

That is, if such tori exist, then using a solution of the first deterministic equation in system (4.14), one can immediately write down a solution of the total stochastic system.

The argument ω will be omitted in the sequel.

Conditions for existence of invariant tori, their stability, dichotomy were obtained for ordinary linear extensions in [139]. It was essential there to use Green's function for the invariant torus as to obtain its integral representation.

In this section, we obtain a similar integral representation for the random invariant torus but with a use of the stochastic Ito integral.

Denote by $\Phi^t_\tau(\varphi)$ a matriciant of the system

$$\frac{dh}{dt} = P(\varphi_t(\varphi))h\,. \tag{4.16}$$

As in [139, p. 12]. introduce Green's function for the problem on the invariant torus. Namely, consider the matrix

$$G_0(\tau,\varphi) = \begin{cases} \Phi^0_\tau(\varphi), & \tau \le 0\,, \\ 0, & \tau > 0\,, \end{cases} \tag{4.17}$$

and call it *Green's function* for the problem on random invariant torus if the integrals

$$\int_{-\infty}^0 ||G_0(\tau,\varphi)||\,d\tau, \qquad \int_{-\infty}^0 ||G_0(\tau,\varphi)||^2\,d\tau$$

uniformly converge in φ. Then

$$\int_{-\infty}^0 ||G_0(\tau,\varphi)||\,d\tau + \int_{-\infty}^0 ||G_0(\tau,\varphi)||^2\,d\tau \le K\,. \tag{4.18}$$

Consider the expression

$$\int_{-\infty}^0 G_0(\tau,\varphi)f(\varphi_\tau(\varphi))\,d\tau + \int_{-\infty}^0 G_0(\tau,\varphi)g(\varphi_\tau(\varphi))\,dW(\tau+t)\,, \tag{4.19}$$

and prove that the integrals exist, the first one as the usual integral, and the second one as a stochastic Ito integral.

Indeed, as it follows from (4.18), the first integral can be majorized with a convergent integral,

$$\left\| \int_{-\infty}^0 G_0(\tau,\varphi)f(\varphi_\tau(\varphi))\,d\tau \right\| \le \int_{-\infty}^0 ||G_0(\tau,\varphi)||\,d\tau \max_{\varphi\in\Im_m} |f(\varphi)|\,.$$

Let us prove that the second integral exists. Making the change of variables $t+\tau = s$ we get

$$\int_{-\infty}^0 G_0(\tau,\varphi)g(\varphi_\tau(\varphi))\,dW(\tau+t) = \int_{-\infty}^t G_0(s-t,\varphi)g(\varphi_{s-t}(\varphi))\,dW(s)\,.$$

Existence of this integral, regarded as the mean square limit of proper stochastic integrals (see Section 3.3) for arbitrary $t \in \mathbf{R}$, follows from the estimate

$$\int_{-\infty}^{t} ||G_0(s-t,\varphi)||^2 |g(\varphi_{s-t}(\varphi))|^2\, ds = \int_{-\infty}^{0} ||G_0(\tau,\varphi)||^2 |g(\varphi_\tau(\varphi))|^2\, d\tau$$
$$\leq \int_{-\infty}^{0} ||G_0(\tau,\varphi)||^2\, d\tau \max_{\varphi\in\Im_m} |g(\varphi)|^2 .$$

It is clear that the first integral in (4.19) is continuous in $\varphi \in \Im_m$.

Let us show that the second integral in this expression is square mean continuous in φ.

Indeed, for arbitrary $n > 0$, it follows from the estimate

$$\mathbf{E}\left| \int_{-n}^{t} G_0(\tau-t,\varphi) g(\varphi_{\tau-t}(\varphi))\, dW(\tau) - \int_{-n}^{t} G_0(\tau-t,\varphi') g(\varphi_{\tau-t}(\varphi'))\, dW(\tau)\right|^2$$
$$= \int_{-n}^{0} ||G_0(\tau,\varphi) g(\varphi_\tau(\varphi)) - G_0(\tau,\varphi') g(\varphi_\tau(\varphi'))||^2\, d\tau \to 0,\ \ \varphi \to \varphi' ,$$

that the stochastic integral

$$\int_{-n}^{0} G_0(\tau,\varphi) g(\varphi_\tau(\varphi))\, dW(\tau+t)$$

is square mean continuous in $\varphi \in \Im_m$ and, hence,

$$\mathbf{E}\left| \int_{-\infty}^{0} G_0(\tau,\varphi) g(\varphi_\tau(\varphi))\, dW(\tau+t) - \int_{-\infty}^{0} G_0(\tau,\varphi') g(\varphi_\tau(\varphi'))\, dW(\tau+t)\right|^2$$
$$\leq 2\left[\mathbf{E}\left| \int_{-n}^{0} G_0(\tau,\varphi) g(\varphi_\tau(\varphi)) - G_0(\tau,\varphi') g(\varphi_\tau(\varphi'))\, dW(\tau+t)\right|^2\right.$$
$$\left. + \,\mathbf{E}\left| \int_{-\infty}^{-n} G_0(\tau,\varphi) g(\varphi_\tau(\varphi)) - G_0(\tau,\varphi') g(\varphi_\tau(\varphi'))\, dW(\tau+t)\right|^2\right]$$
$$\leq 2\mathbf{E}\left| \int_{-n}^{0} (G_0(\tau,\varphi) g(\varphi_\tau(\varphi)) - G_0(\tau,\varphi') g(\varphi_\tau(\varphi'))\, dW(\tau+t)\right|^2$$
$$+ \,8 \int_{-\infty}^{-n} ||G_0(\tau,\varphi)||^2\, d\tau \sup_{\varphi\in\Im_m} |g(\varphi)|^2 .$$

The first term in the last inequality approaches zero as $\varphi \to \varphi'$, and the second one tends to zero uniformly in $\varphi \in \Im_m$ as $n \to \infty$.

So, the first and the second terms in (4.19) are square mean continuous, so the whole expression in (4.19) is square mean continuous.

Remark. If the function $g(\varphi)$ is Lipschitz continuous with respect to φ, $a(\varphi) \in C^1(\Im_m)$, and $G_0(t,\varphi)$ is differentiable in φ and satisfies the estimate

$$\left\|\frac{\partial G_0(t,\varphi)}{\partial \varphi_i}\right\| \leq K\exp\{\gamma t\},\ i = 1\ldots m,\ t \leq 0\,,$$

(conditions for the estimate to hold can be found in [139, p. 192]), then the torus defined by (4.19) is continuous in φ with probability 1.

Indeed, we have

$$\begin{aligned}
&\mathbf{E}\left|\int_{-\infty}^{0}[G_0(\tau,\varphi)g(\varphi_\tau(\varphi)) - G_0(\tau,\varphi')g(\varphi_\tau(\varphi'))]\,dW(\tau+t)\right|^2\\
&\leq \int_{-\infty}^{0}|G_0(\tau,\varphi)g(\varphi_\tau(\varphi)) - G_0(\tau,\varphi')g(\varphi_\tau(\varphi'))|^2\,d\tau\\
&\leq 2\left[\int_{-\infty}^{0}||G_0(\tau,\varphi)||^2|g(\varphi_\tau(\varphi)) - g(\varphi_\tau(\varphi'))|^2\,d\tau\right.\\
&\qquad\left.+\int_{-\infty}^{0}||G_0(\tau,\varphi) - G_0(\tau,\varphi')||^2|g(\varphi_\tau(\varphi'))|^2\,d\tau\right],
\end{aligned}$$

which shows, since $\varphi_t(\varphi)$ is smooth in $\varphi \in \Im_m$ regarded as a parameter, that the latter inequality can be continued to

$$2\left[L\int_{-\infty}^{0}||G_0(\tau,\varphi)||^2\,d\tau|\varphi-\varphi'|+\int_{-\infty}^{0}K^2\exp\{\gamma\tau\}\,d\tau L_1|\varphi-\varphi'|\right] \leq C|\varphi-\varphi'|^2\,.$$

Now, Kolmogorov's theorem on continuity of random processes proves the claim in the remark.

Let us, finally, show that the expression

$$\begin{aligned}
h = u(t,\varphi,\omega) &= \int_{-\infty}^{0}G_0(\tau,\varphi)f(\varphi_\tau(\varphi))\,d\tau\\
&\quad+\int_{-\infty}^{0}G_0(\tau,\varphi)g(\varphi_\tau(\varphi))\,dW(\tau+t) \qquad (4.20)
\end{aligned}$$

defines an invariant torus for system (4.14).

To this end, we verify that all conditions in Definition 4.2 are satisfied.

Since, for arbitrary $\varphi \in \Im_m$, $t \in \mathbf{R}$, the stochastic integral in (4.20) is a random variable, formula (4.20) defines a random function periodic in φ. Its periodicity in φ_i follows since the integrands are 2π-periodic in φ_i.

Let us show that the function

$$h_t = u(t, \varphi_t(\varphi), \omega) = \int_{-\infty}^{0} G_0(\tau, \varphi_t(\varphi)) f(\varphi_\tau(\varphi_t(\varphi)))\, d\tau$$
$$+ \int_{-\infty}^{0} G_0(\tau, \varphi_t(\varphi)) g(\varphi_\tau(\varphi_t(\varphi)))\, dW(\tau + t) \qquad (4.21)$$

is a solution of system (4.15).

Indeed, using the properties of Green's function we have

$$h_t = \int_{-\infty}^{0} G_t(\tau + t, \varphi) f(\varphi_{\tau+t}(\varphi))\, d\tau + \int_{-\infty}^{0} G_t(\tau + t, \varphi) g(\varphi_{\tau+t}(\varphi))\, dW(\tau + t)$$
$$= \int_{-\infty}^{t} G_t(\tau, \varphi) f(\varphi_\tau(\varphi))\, d\tau + \int_{-\infty}^{t} G_t(\tau, \varphi) g(\varphi_\tau(\varphi))\, dW(\tau)$$

for $t \in \mathbf{R}$. This representation and the properties of a stochastic integral yield that h_t is F_t-measurable for arbitrary $t \in \mathbf{R}$.

It is clear that conditions 2) and 3) of Definition 4.1 are satisfied. Let us show that condition 4) in this definition is also satisfied.

To do this, using Lemma 3.2 let us evaluate the stochastic differential dh_t. We have

$$dh_t = \left(\int_{-\infty}^{t} \frac{\partial G_t}{\partial t}(\tau, \varphi) f(\varphi_\tau(\varphi)) d\tau + G_t(t, \varphi) f(\varphi_t(\varphi)) \right) dt$$
$$+ \int_{-\infty}^{t} \frac{\partial G_t}{\partial t}(\tau, \varphi) g(\varphi_\tau(\varphi))\, dW(\tau) dt + G_t(t, \varphi) g(\varphi_t(\varphi)) dW(t)\,. \quad (4.22)$$

Since $G_t(\tau, \varphi) = \Phi_\tau^t(\varphi)$ for $t \geq \tau$, by [139, p. 122] we have

$$\frac{\partial G_t(\tau, \varphi)}{\partial t} = P(\varphi_t(\varphi)) G_t(\tau, \varphi), \;\; G_t(t, \varphi) = E$$

and so, by continuity and periodicity in $P(\varphi)$, the integrals obtained by formal differentiation converge uniformly in $t \in \mathbf{R}$, since

$$\int_{-\infty}^{t} ||G_t(\tau, \varphi)||\, d\tau = \int_{-\infty}^{0} ||G_t(\tau + t, \varphi)||\, d\tau = \int_{-\infty}^{0} ||G_0(\tau, \varphi_t(\varphi))||\, d\tau\,.$$

However, $\varphi_t(\varphi) \in \Im_m$ for arbitrary $t \in \mathbf{R}$ and $\varphi \in \Im_m$ and so, by the definition of Green's function, the last integral converges uniformly with respect to $\varphi \in \Im_m,\; t \in \mathbf{R}$.

We have proved formula (4.22). It shows that

$$
\begin{aligned}
dh_t &= P(\varphi_t(\varphi))\left[\int_{-\infty}^{t} G_t(\tau,\varphi)f(\varphi_\tau(\varphi))\,d\tau + \int_{-\infty}^{t} G_t(\tau,\varphi)g(\varphi_\tau(\varphi))\,dW(\tau)\right] \\
&\quad + f(\varphi_t(\varphi)))dt + g(\varphi_t(\varphi))dW(t) \\
&= (P(\varphi_t(\varphi))h_t + f(\varphi_t(\varphi)))dt + g(\varphi_t(\varphi))dW(t)\,,
\end{aligned}
$$

hence, $h_t = u(t, \varphi_t(\varphi), \omega)$ is a solution of equation (4.15). Thus, expression (4.20) defines an invariant torus for system (4.14).

Let us show that the torus $h = u(t, \varphi, \omega)$ is mean square bounded. We have

$$
\begin{aligned}
\sup_{t\in\mathbf{R},\varphi\in\Im_m} \mathbf{E}|u(t,\varphi,\omega)|^2 &\le 2\sup_{\varphi\in\Im_m}\left(\int_{-\infty}^{0}||G_0(\tau,\varphi)||\,d\tau\right)^2 \sup_{\varphi\in\Im_m}|f(\varphi)|^2 \\
&\quad + 2\sup_{t\in\mathbf{R},\varphi\in\Im_m}\mathbf{E}\left|\int_{-\infty}^{t} G_0(s-t,\varphi)g(\varphi_{s-t}(\varphi))\,dW(s)\right|^2 \\
&= K_1\sup_{\varphi\in\Im_m}|f(\varphi)|^2 + 2\sup_{t\in\mathbf{R},\varphi\in\Im_m}\int_{-\infty}^{t}||G_0(s-t,\varphi)||^2|g(\varphi_{s-t}(\varphi))|^2\,ds \\
&= K_1\sup_{\varphi\in\Im_m}|f(\varphi)|^2 + 2\sup_{\varphi\in\Im_m}\int_{-\infty}^{0}||G_0(\tau,\varphi)||^2|g(\varphi_\tau(\varphi))|^2\,d\tau \\
&\le K_1\sup_{\varphi\in\Im_m}|f(\varphi)|^2 + K_2\sup_{\varphi\in\tau_m}|g(\varphi)|^2\,,
\end{aligned}
\tag{4.23}
$$

where

$$
K_1 = 2\sup_{\varphi\in\Im_m}\left(\int_{-\infty}^{0}||G_0(\tau,\varphi)||\,d\tau\right)^2, \qquad K_2 = 2\sup_{\varphi\in\Im_m}\int_{-\infty}^{0}||G_0(\tau,\varphi)||^2\,d\tau\,.
$$

The following theorem is a corollary of the above.

Theorem 4.3. *If the right-hand side of the system*

$$
\frac{d\varphi}{dt} = a(\varphi), \qquad \frac{dh}{dt} = P(\varphi)h
$$

is Lipschitz continuous in φ, $a(\varphi) \in C(\Im_m)$, *and* $P(\varphi) \in C(\Im_m)$ *has Green's function* $G_0(\tau,\varphi)$ *for* (4.17), *then, for arbitrary* $f(\varphi)$, $g(\varphi) \in C(\Im_m)$, *system* (4.14) *has a random invariant torus defined by formula* (4.20), *and it satisfies estimate* (4.23).

Example. Consider the equation

$$d\varphi = -\sin\varphi dt, \qquad dh = (-h + \sin\varphi)dt + \sin\varphi dW(t)\,. \tag{4.24}$$

Here $\varphi \in [0,\ 2\pi]$, $h \in \mathbf{R}$, $W(t)$ is a one-dimensional Wiener process on $\mathbf{R}$. By (4.17), this system has Green's function. It is given by

$$G_0(\tau,\varphi) = \begin{cases} \exp\{\tau\}, & \tau \le 0\,, \\ 0, & \tau > 0\,. \end{cases}$$

It then follows from (4.20) that the invariant torus is given by

$$h = \int_{-\infty}^{0} \exp\{\tau\} \sin\varphi_\tau(\varphi)\, d\tau + \int_{-\infty}^{0} \exp\{\tau\} \sin\varphi_\tau(\varphi)\, dW(t+\tau)\,,$$

where

$$\sin(\varphi_\tau(\varphi)) = \begin{cases} 0, & \varphi = k\pi, k \in Z\,, \\ \dfrac{2\exp\{\tau\}\tan(\frac{\varphi}{2})}{\exp\{2\tau\} + \tan^2(\frac{\varphi}{2})}, & \varphi \ne k\pi\,. \end{cases}$$

A calculation for $h = u(t,\varphi,\omega)$ gives that

$$h = u(t,\varphi,\omega) = \begin{cases} 0, & \varphi = k\pi, k \in Z, \\ \tan\left(\dfrac{\varphi}{2}\right) \ln\left(\sin^2\left(\dfrac{\varphi}{2}\right)\right) + 2\exp\{-t\}\tan\left(\dfrac{\varphi}{2}\right) & \\ \times \displaystyle\int_{-\infty}^{t} \frac{\exp\{2s\}}{\exp\{2(s-t)\} + \tan^2(\frac{\varphi}{2})}\, dW(s), & \varphi \ne k\pi\,, \end{cases}$$

which defines a random invariant torus for system (4.14).

Let us remark that Theorem 4.3 connects the conditions for existence of invariant tori for system (4.14) with existence of Green's function for the problem on random invariant tori for the homogeneous deterministic system

$$\frac{d\varphi}{dt} = a(\varphi), \qquad \frac{dh}{dt} = P(\varphi)h. \tag{4.25}$$

The latter is closely related to stability of this system. This is so, if the matriciant of system (4.16) admits the estimate

$$||\Phi_\tau^t(\varphi)|| \le K\exp\{-\gamma(t-\tau)\} \tag{4.26}$$

for $t \ge \tau$ with some positive constants K and γ independent of t, τ, and φ.

In particular, this is true if the matrix $P(\varphi)$ satisfies the inequality

$$(P(\varphi)h, h) \leq -\gamma |h|^2 \tag{4.27}$$

for arbitrary $h \in \mathbf{R}^n$, $\varphi \in \Im_m$. Indeed, in such a case, we see that, by the Vazhevsky inequality, the solution h_t of system (4.16) satisfies the following condition for arbitrary $\varphi \in \Im_m$:

$$\frac{d}{dt}(h_t, h_t) \leq -2\gamma |h_t|^2 \,,$$

that is,

$$|h_t| \leq \exp\{-\gamma(t-\tau)\}|h_\tau|$$

for $t > \tau$. The latter inequality holds for an arbitrary solution of system (4.16), hence for any column of the matriciant too, which leads to inequality (4.26).

Let us show that if estimate (4.26) holds, then the invariant torus for system (4.14) is mean square exponentially stable with probability 1.

To this end, make a change of variables in (4.14),

$$h = u(t, \varphi, \omega) + z \,. \tag{4.28}$$

We see that if $(\varphi_t(\varphi), h_t)$ is a solution of system (4.14), then

$$\begin{aligned} dh_t = du(t, \varphi_t, \omega) + dz_t = (P(\varphi_t(\varphi))(u(t, \varphi_t(\varphi), \omega) + z_t) \\ + f(\varphi_t(\varphi)))dt + g(\varphi_t(\varphi))dW(t) \,, \end{aligned}$$

which shows that

$$dz_t = P(\varphi_t(\varphi)) z_t dt.$$

Hence, z_t is a solution of the linear homogeneous differential system and, by (4.26), we have that

$$|z_t| \leq K \exp\{-\gamma(t-\tau)\}|z_\tau|$$

or

$$|h_t - u(t, \varphi_t(\varphi), \omega)| \leq K \exp\{-\gamma(t-\tau)\}|h_\tau - u(\tau, \varphi, \omega)| \,.$$

The latter inequality shows that the torus is exponentially stable with probability 1. Moreover, since h_τ and $u(\tau, \varphi, \omega)$ have finite second moments for arbitrary $\tau \in \mathbf{R}$, the random invariant torus is mean square exponentially stable.

Let us remark that, by [139, p. 126], for existence of an exponentially stable invariant torus for system (4.14), it is sufficient that a condition weaker than inequality (4.26) be satisfied, namely,

$$||\Phi_0^t(\varphi)|| \leq K \exp\{-\gamma t\} \tag{4.29}$$

for $t \geq 0$.

Summing up the above results gives the following theorem.

Theorem 4.4. *1) If inequality* (4.29) *is satisfied, then system* (4.14) *has a random invariant torus defined by relations* (4.20)*. It is exponentially stable with probability* 1 *and mean square exponentially stable.*

2) For inequality (4.29) *to hold, it is sufficient that the matrix* $P(\varphi)$ *would satisfy inequality* (4.27)*.*

From the results of [139, pp. 127–130], it follows that the conditions in the above theorem can be reformulated in terms of quadratic forms. This leads, in particular, to the following.

Theorem 4.5. *For system* (4.14) *to have a random invariant torus, which is exponentially stable with probability* 1 *and mean square exponentially stable and is given by formula* (4.20)*, it is necessary and sufficient that there existed a positive definite symmetric matrix* $S(\varphi) \in C^1(\Im_m)$ *such that the matrix*

$$\frac{\partial S(\varphi)}{\partial \varphi} a(\varphi) + P^T(\varphi) S(\varphi) + S(\varphi) P(\varphi)$$

is negative definite.

Example. Set $S(\varphi) = S = const$. Then the conditions of Theorem 4.5 are satisfied if the eigen values of the matrix S are positive, and the eigen values of the matrix $P^T(\varphi)S + SP(\varphi)$ are negative.

4.3 Smoothness of invariant tori

Let us now consider the dependence between smoothness of the invariant torus for system (4.14) and smoothness of its right-hand side. As it is known, such a dependence is not evident even in the deterministic case. In the example given after Theorem 4.3, the torus is only mean square continuous in φ, regardless that the coefficients in the right-hand side of system (4.24) are analytic in φ.

Definition 4.3. A random function $\frac{\partial u(t,\varphi_0,\omega)}{\partial \varphi_i}$ is called *mean square partial derivative* of a random torus $h = u(t,\varphi,\omega)$ with respect to $\varphi_i (i = 1 \ldots m)$ in a point φ_0 if

$$\lim_{\varphi_i \to 0} \mathbf{E}| \frac{u(t,\varphi_{01},\ldots \varphi_{0i}+\varphi_i,\ldots \varphi_{0m},\omega) - u(t,\varphi_{01},\ldots \varphi_{0i},\ldots \varphi_{0m},\omega)}{\varphi_i}$$
$$- \frac{\partial u(t,\varphi_0,\omega)}{\partial \varphi_i}|^2 = 0\,. \tag{4.30}$$

Theorem 4.6. *If, in system* (4.14), *a, P, f, g* $\in C^1(\Im_m)$ *and the matriciant* $\Phi_0^t(\varphi)$ *for system* (4.16) *satisfies inequality* (4.29), *then assuming that the inequality* $\gamma > \alpha$ *holds, where*

$$\alpha = \max_{\varphi \in \Im_m} \left\| \frac{\partial a(\varphi)}{\partial \varphi} \right\|,$$

the random torus is mean square continuously differentiable in φ.

Its mean square derivative is given by the formula

$$\frac{\partial u}{\partial \varphi_i} = \int_{-\infty}^0 \frac{\partial G_0(\tau,\varphi)}{\partial \varphi_i} f(\varphi_\tau(\varphi))\,d\tau + \int_{-\infty}^0 G_0(\tau,\varphi) \frac{\partial f(\varphi_\tau(\varphi))}{\partial \varphi_i} \frac{\partial \varphi_\tau(\varphi)}{\partial \varphi_i}\,d\tau$$
$$+ \int_{-\infty}^0 \frac{\partial G_0(\tau,\varphi)}{\partial \varphi_i} g(\varphi_\tau(\varphi))\,dW(t+\tau)$$
$$+ \int_{-\infty}^0 G_0(\tau,\varphi) \frac{\partial g(\varphi_\tau(\varphi))}{\partial \varphi_i} \frac{\partial \varphi_\tau(\varphi)}{\partial \varphi_i}\,dW(t+\tau)\,. \tag{4.31}$$

Proof. Formal differentiation of the integral representation (4.20) for the torus gives the right-hand side of (4.31).

Let us find conditions for the integrals to make sense. Since $a, P, f, g, \in C^1(\Im_m)$, for the first two integrals to exist and be differentiable with respect to the parameter, it is sufficient that they were uniformly convergent. Let us make an estimate for the integrands, taking into account that Green's function, under condition (4.29), satisfies estimate (4.26).

First, consider the second integral in (4.31). Since $f \in C^1(\Im_m)$, we see that $\frac{\partial f(\varphi)}{\partial \varphi_i}$ is bounded on the torus $\Im_m$ and, hence, this integral is majorized by the integral

$$\int_{-\infty}^0 K \exp\{\gamma\tau\} \left| \frac{\partial \varphi_\tau(\varphi)}{\partial \varphi_i} \right| d\tau\,. \tag{4.32}$$

The function $\frac{\partial \varphi_\tau(\varphi)}{\partial \varphi_i}$ is a solution of the linear system in variations,

$$\frac{d}{dt} \frac{\partial \varphi_t(\varphi))}{\partial \varphi} = \frac{\partial a(\varphi_t(\varphi))}{\partial \varphi} \frac{\partial \varphi_t(\varphi)}{\partial \varphi},$$

so finding its estimate we have

$$\left\|\frac{\partial\varphi_t(\varphi)}{\partial\varphi}\right\| \leq K_1 \exp\left\{\max_{\varphi\in\Im_m}\left\|\frac{\partial a(\varphi)}{\partial\varphi}\right\||t|\right\}$$

for $t \in \mathbf{R}$.

The latter inequality shows that the second integral in (4.31) is majorized with the integral

$$\int_{-\infty}^{0} K_2 \exp\{(\gamma-\alpha)\tau\}\,d\tau \tag{4.33}$$

and, hence, it converges uniformly in $\varphi \in \Im_m$.

Let us now find an estimate for the first integral in (4.31). We have

$$\left\|\frac{\partial G_0(\tau,\varphi)}{\partial\varphi_i} f(\varphi_\tau(\varphi))\right\| \leq K_3 \left\|\frac{\partial\Phi_\tau^0(\varphi)}{\partial\varphi_i}\right\|.$$

However, as before, the derivative of the matriciant again satisfies the equations in variations,

$$\frac{d}{dt}\left(\frac{\partial\Phi_\tau^t(\varphi)}{\partial\varphi_i}\right) = P(\varphi_t(\varphi))\frac{\partial\Phi_\tau^t(\varphi)}{\partial\varphi_i} + \frac{\partial P(\varphi_t(\varphi))}{\partial\varphi}\frac{\partial\varphi_t(\varphi)}{\partial\varphi_i}\Phi_\tau^t(\varphi).$$

From the Cauchy formula, it follows that

$$\begin{aligned}\left\|\frac{\partial\Phi_\tau^t(\varphi)}{\partial\varphi_i}\right\| &\leq ||\Phi_\tau^t(\varphi)|| + \int_\tau^t \left\|\Phi_s^t(\varphi)\frac{\partial P(\varphi_s(\varphi))}{\partial\varphi_i}\frac{\partial\varphi_s(\varphi)}{\partial\varphi_i}\Phi_\tau^s(\varphi)\right\| ds \\ &\leq K\exp\{-\gamma(t-\tau)\} + K_5\exp\{-\gamma(t-\tau)\}\int_\tau^t \exp\{\alpha(s-\tau)\}\,ds \\ &\leq K\exp\{-\gamma(t-\tau)\} + \frac{K_5}{\alpha}[\exp\{-(\gamma-\alpha)(t-\tau)\} - \exp\{-\gamma(t-\tau)\}]\,,\end{aligned} \tag{4.34}$$

which proves that the first integral in (4.31) converges uniformly in $\varphi \in \Im_m$, hence, it can be differentiated with respect to the parameter.

Let us now consider stochastic integrals in formula (4.31). Using estimates (4.26), (4.33), and (4.34) we see that these integrals exist, since the integrals

$$\int_{-\infty}^{0}\left\|\frac{\partial G_0(\tau,\varphi)}{\partial\varphi_i}\right\|^2 d\tau, \qquad \int_{-\infty}^{0}||G_0(\tau,\varphi)||^2\left\|\frac{\partial\varphi_{\tau(\varphi)}}{\partial\varphi_i}\right\|^2 d\,\tau \tag{4.35}$$

converge uniformly in $\varphi \in \Im_m$.

Let us finally show that the two last expressions in (4.31) give the mean square derivative with respect to φ of the stochastic part in (4.20) in the sense of Definition 4.3. We have

$$\mathbf{E}\bigg\|\int_{-\infty}^{0}\bigg(\frac{G_0(\tau,\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m,)g(\varphi_\tau(\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m))}{\varphi_i}$$

$$-\frac{G_0(\tau,\varphi_0)g(\varphi_\tau(\varphi_0))}{\varphi_i}-\frac{\partial G_0(\tau,\varphi_0)}{\partial\varphi_i}g(\varphi_\tau(\varphi_0))-G_0(\tau,\varphi_0)\frac{\partial g(\varphi_\tau(\varphi_0))}{\partial\varphi_i}$$

$$\times\frac{\partial\varphi_\tau(\varphi_0)}{\partial\varphi_i}\bigg)dW(t+\tau)\bigg\|^2$$

$$=\int_{-\infty}^{0}\bigg\|\frac{G_0(\tau,\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m,)g(\varphi_\tau(\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m))}{\varphi_i}$$

$$-\frac{G_0(\tau,\varphi_0)g(\varphi_\tau(\varphi_0))}{\varphi_i}-\frac{\partial G_0(\tau,\varphi_0)}{\partial\varphi_i}g(\varphi_\tau(\varphi_0))$$

$$-G_0(\tau,\varphi_0)\frac{\partial g(\varphi_\tau(\varphi_0))}{\partial\varphi_i}\frac{\partial\varphi_\tau(\varphi_0)}{\partial\varphi_i}\bigg\|^2 d\tau\,. \tag{4.36}$$

The integrand in (4.36) tends to zero as $\varphi_i \to 0$ uniformly in $\tau \in [0,\ -A]$ for every $A > 0$, $\varphi_0 \in \Im_m$, since the derivative $\frac{\partial}{\partial\varphi_i}(G_0(\tau,\varphi_0)g(\varphi_\tau(\varphi_0)))$ is uniformly continuous in $\tau \in [0,\ -A], \varphi_0 \in \Im_m$, and due to Lagrange's formula for representing the difference relation in (4.36). Since the integrals in (4.35) converge uniformly in $\varphi \in \Im_m$, one can pass to the limit with respect to φ_i in the integrand in (4.36). This leads to the final relations

$$\mathbf{E}\bigg\|\frac{u(t,\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m,\omega)-u(t,\varphi_0,\omega)}{\varphi_i}\bigg\|^2$$

$$=\mathbf{E}\bigg\|\int_{-\infty}^{0}\bigg(\frac{G_0(\tau,\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m)f(\varphi_\tau(\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m))}{\varphi_i}$$

$$-\frac{G_0(\tau,\varphi_0)f(\varphi_\tau(\varphi_0))}{\varphi_i}-\frac{\partial}{\partial\varphi_i}(G_0(\tau,\varphi_0)f(\varphi_\tau(\varphi_0)))\bigg)d\tau$$

$$+\int_{-\infty}^{0}\bigg(\frac{G_0(\tau,\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m,)g(\varphi_\tau(\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m))}{\varphi_i}$$

$$-\frac{G_0(\tau,\varphi_0)g(\varphi_\tau(\varphi_0))}{\varphi_i}-\frac{\partial}{\partial\varphi_i}(G_0(\tau,\varphi_0)g(\varphi_\tau(\varphi_0)))\bigg)dW(t+\tau)\bigg\|^2$$

$$\le 2\bigg[\bigg\|\int_{-\infty}^{0}\bigg(\frac{G_0(\tau,\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m)f(\varphi_\tau(\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m))}{\varphi_i}$$

$$- \frac{G_0(\tau,\varphi_0)f(\varphi_\tau(\varphi_0))}{\varphi_i} - \frac{\partial}{\partial\varphi i}(G_0(\tau,\varphi_0)f(\varphi_\tau(\varphi_0)))\Big) d\tau\Big\|^2$$

$$+ \int_{-\infty}^0 \Big\| \frac{G_0(\tau,\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m,)g(\varphi_\tau(\varphi_{01},\dots\varphi_{0i}+\varphi_i,\dots\varphi_m))}{\varphi_i}$$

$$- \frac{G_0(\tau,\varphi_0)g(\varphi_\tau(\varphi_0))}{\varphi_i} - \frac{\partial}{\partial\varphi_i}(G_0(\tau,\varphi_0)g(\varphi_\tau(\varphi_0)))\Big\|^2 d\tau\Big] \to 0\,,$$

which prove existence of a mean square derivative of the form (4.31), which finishes the proof. □

By making similar calculations for higher order derivatives, we get the following result.

Corollary 4.1. *Let, in system* (4.14), $a, P, f, g \in C^l(\Im_m)$ *and the matriciant* $\Phi_0^t(\varphi)$ *for system* (4.16) *satisfy inequality* (4.29). *If the inequality* $\gamma > l\alpha$ *is satisfied, the random torus* (4.20) *for system* (4.14) *is* l *times mean square continuously differentiable in* φ.

4.4 Random invariant tori for nonlinear extensions

Let us now consider random invariant tori for nonlinear stochastic systems of the form

$$d\varphi = a(\varphi)dt,\ dh = (P(\varphi)h + A(\varphi,h,\varepsilon))dt + \ Q(\varphi,h,\varepsilon)dW(t)\,, \tag{4.37}$$

where ε is a small positive parameter and

$$A(\varphi,0,0) = Q(\varphi,0,0) = 0\,. \tag{4.38}$$

The latter condition implies that system (4.37) has the trivial invariant torus $h = 0$, $\varphi \in \Im_m$, for $\varepsilon = 0$. Let the functions a, P, A, Q be jointly continuous for $\varphi \in \Im_m,\ h \in \mathbf{R}^n$, $\varepsilon \in [0,\ \varepsilon_0]$, periodic in φ_i $(i = 1,\dots m)$ with period 2π, the function a be Lipschitz continuous in φ, the functions A, Q be Lipschitz continuous in h with the Lipschitz constant $L(\varepsilon) \to 0$ as $\varepsilon \to 0$. These conditions, together with (4.38), imply that there are functions $\alpha(\varepsilon) \to 0$, $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$|A(\varphi,h,\varepsilon)|^2 \le 2L^2|h|^2 + \alpha(\varepsilon), |Q(\varphi,h,\varepsilon)|^2 \le 2L^2|h|^2 + \beta(\varepsilon)\,. \tag{4.39}$$

Denote $2L^2(\varepsilon) = N(\varepsilon)$. Let us write a system in variations (4.37) corresponding to the torus $h = 0,\ \varphi \in \Im_m$,

$$d\varphi = a(\varphi)dt, \qquad dh = P(\varphi)hdt\,. \tag{4.40}$$

The following theorem gives conditions for existence of an invariant torus for system (4.37).

Theorem 4.7. *Let the right-hand sides of system* (4.37) *satisfy the above conditions. If system* (4.40) *has Green's function* $G_0(\tau, \varphi)$ *that satisfies the estimate*

$$||G_0(\tau, \varphi)|| \le K \exp\{\gamma\tau\}, \qquad \tau < 0\,, \tag{4.41}$$

where $K > 0$, $\gamma > 0$ *are constants independent of* τ, φ, *then there exists* $0 < \varepsilon_1 \le \varepsilon_0$ *such that system* (4.37) *has a random invariant torus* $h = u(t, \varphi, \varepsilon)$ *for arbitrary* $\varepsilon \in (0,\ \varepsilon_1]$.

Proof. Denote by B the Banach space of n-dimensional random functions $\xi(t, \varphi, \omega)$ defined for $t \in \mathbf{R}, \varphi \in \Im_m, \omega \in \Omega$, jointly measurable, F_t-measurable for every t, φ, periodic with probability 1 in φ_i with period 2π, and endowed with the norm

$$||\xi||_2 = \left(\sup_{t \in \mathbf{R}, \varphi \in \Im_m} \mathbf{E}|\xi(t, \varphi, \omega)|^2 \right)^{\frac{1}{2}}.$$

Define an operator S on the space B by

$$\begin{aligned} Su = &\int_{-\infty}^{0} G_0(\tau, \varphi) A(\varphi_\tau(\varphi), u(\tau + t, \varphi_\tau(\varphi)), \varepsilon)\, d\tau \\ &+ \int_{-\infty}^{0} G_0(\tau, \varphi) Q(\varphi_\tau(\varphi), u(\tau + t, \varphi_\tau(\varphi)), \varepsilon)\, dW(t + \tau)\,. \end{aligned} \tag{4.42}$$

Let us show that $S : B \to B$. To this end, we first show that both integrals exist. Denote them by I_1 and I_2, correspondingly. Existence of the integral I_1 is implied by the Fubini theorem, since estimates (4.39) and (4.41) give

$$\begin{aligned} &\int_{-\infty}^{0} ||G_0(\tau, \varphi)|| \mathbf{E}|A(\varphi_\tau(\varphi), u(\tau + t, \varphi_\tau(\varphi)), \varepsilon)|\, d\tau \\ &\le \int_{-\infty}^{0} K \exp\{\gamma\tau\} (L(\varepsilon)\mathbf{E}|u(\tau + t, \varphi_\tau(\varphi))| + |A(\varphi_\tau(\varphi), 0, \varepsilon)|)\, d\tau \\ &\le \int_{-\infty}^{0} K \exp\{\gamma\tau\} L(\varepsilon)\, d\tau ||u(t, \varphi)||_2 + \int_{-\infty}^{0} K \exp\{\gamma\tau\} |A(\varphi_\tau(\varphi), 0, \varepsilon)|\, d\tau < \infty\,. \end{aligned}$$

It also follows from (4.39) and (4.41) that

$$\int_{-\infty}^{0} ||G_0(\tau,\varphi)||^2 \mathbf{E}|Q(\varphi_\tau(\varphi), u(\tau+t,\varphi_\tau(\varphi)),\varepsilon)|^2 \, d\tau$$
$$\leq \int_{-\infty}^{0} K^2 \exp\{2\gamma\tau\}(\beta(\varepsilon) + N(\varepsilon)||u(t,\varphi)||_2^2)\, d\tau \leq \infty \,,$$

and, hence, estimate (3.66) in Lemma 3.2 is verified. So, the two integrals in (4.42) exist with probability 1. Now, we have

$$\begin{aligned}
\mathbf{E}|I_1|^2 &\leq \mathbf{E}\bigg(\int_{-\infty}^{0} K \exp\{\gamma\tau\}|A(\varphi_\tau(\varphi), u(\tau+t,\varphi_\tau(\varphi)),\varepsilon)|\, d\tau\bigg)^2 \\
&\leq K^2 \int_{-\infty}^{0} \exp\{\gamma\tau\}\, d\tau \int_{-\infty}^{0} \exp\{\gamma\tau\}\mathbf{E}|A(\varphi_\tau(\varphi), u(\tau+t,\varphi_\tau(\varphi)),\varepsilon)|^2\, d\tau) \\
&\leq \frac{K^2}{\gamma} \int_{-\infty}^{0} \exp\{\gamma\tau\}(N(\varepsilon)||u(t,\varphi)||_2^2 + \alpha(\varepsilon))\, d\tau \\
&= \frac{K^2}{\gamma^2}(N(\varepsilon)||u(t,\varphi)||_2^2 + \alpha(\varepsilon))\,.
\end{aligned}$$

Using properties of stochastic integrals we have

$$\begin{aligned}
\mathbf{E}|I_2|^2 &\leq \int_{-\infty}^{0} K^2 \exp\{2\gamma\tau\}\mathbf{E}|Q(\varphi_\tau(\varphi), u(\tau+t,\varphi_\tau(\varphi)),\varepsilon)|^2\, d\tau \\
&\leq \frac{K^2}{2\gamma}(N(\varepsilon)||u(t,\varphi)||_2^2 + \beta(\varepsilon))\,.
\end{aligned}$$

This implies that the integrals in (4.42) are F_t-measurable. So, the operator S maps the Banach space B into itself.

Let u_1, u_2 be arbitrary elements of B. We have

$$\begin{aligned}
\mathbf{E}|Su_1 - Su_2|^2 &\leq \mathbf{E}\bigg(\int_{-\infty}^{0} ||G_0(\tau,\varphi)||L(\varepsilon)|u_1(t+\tau,\varphi_\tau(\varphi)) - u_2(t+\tau,\varphi_\tau(\varphi))|\, d\tau \\
&\quad + \bigg|\int_{-\infty}^{0} G_0(\tau,\varphi)(Q(\varphi_\tau(\varphi), u_1(\tau+t,\varphi_\tau(\varphi)),\varepsilon) \\
&\quad - Q(\varphi_\tau(\varphi), u_2(\tau+t,\varphi_\tau(\varphi)),\varepsilon))dW(t+\tau)\bigg|\bigg)^2 \\
&\leq 2\bigg(\int_{-\infty}^{0} K^2 \exp\{\gamma\tau\}L^2(\varepsilon)\bigg)d\tau
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{-\infty}^{0} \exp\{\gamma\tau\}\mathbf{E}|u_1(t+\tau,\varphi_\tau(\varphi)) - u_2(t+\tau,\varphi_\tau(\varphi))|^2\, d\tau \\
&+ \int_{-\infty}^{0} K^2 \exp\{2\gamma\tau\} L^2(\varepsilon)\mathbf{E}|u_1(t+\tau,\varphi_\tau(\varphi)) \\
&- u_2(t+\tau,\varphi_\tau(\varphi))|^2\, d\tau) \\
\leq{}& \left(\frac{2K^2L^2(\varepsilon)}{\gamma^2} + \frac{K^2}{2\gamma}L^2(\varepsilon)\right)||u_1 - u_2||_2^2\,. \qquad (4.43)
\end{aligned}
$$

Choose $\varepsilon_1 \leq \varepsilon_0$ satisfying the inequality

$$
\left(\frac{2K^2}{\gamma^2} + \frac{K^2}{\gamma}\right)L^2(\varepsilon) < 1\,.
$$

Then it follows from (4.43) that the operator S is a contraction on the Banach space B. Thus it has a fixed point $u(t,\varphi)$ that is a solution of the equation

$$
\begin{aligned}
u(t,\varphi) ={}& \int_{-\infty}^{0} G_0(\tau,\varphi)A(\varphi_\tau(\varphi), u(\tau+t,\varphi_\tau(\varphi)),\varepsilon)\, d\tau \\
&+ \int_{-\infty}^{0} G_0(\tau,\varphi)Q(\varphi_\tau(\varphi), u(\tau+t,\varphi_\tau(\varphi)),\varepsilon)\, dW(t+\tau)\,. \qquad (4.44)
\end{aligned}
$$

Let us show that this random function $u(t,\varphi)$ is an invariant torus for system (4.37). To do this, it is necessary to prove that $h_t = u(t,\varphi_t(\varphi))$ gives a solution of the second equation of system (4.37) for $\varphi = \varphi_t(\varphi)$. It is clear that $u(t,\varphi_t(\varphi))$ is F_t-measurable. Replace the function φ with $\varphi_t(\varphi)$ in (4.44), and take the stochastic differential of both sides. It follows from properties of Green's function and estimate (4.41) that differentiation of the integral I_1 is permitted, and we can apply Lemma 3.2 to the stochastic integral I_2. This leads to the following:

$$
\begin{aligned}
du(t,\varphi_t(\varphi)) ={}& d\Big(\int_{-\infty}^{0} G_0(\tau,\varphi_t(\varphi))A(\varphi_\tau(\varphi_t(\varphi)), u(\tau+t,\varphi_\tau(\varphi_t(\varphi))),\varepsilon)\, d\tau \\
&+ \int_{-\infty}^{0} G_0(\tau,\varphi_t(\varphi))Q(\varphi_\tau(\varphi_t(\varphi)), u(\tau+t,\varphi_\tau(\varphi_t(\varphi))),\varepsilon)\, dW(t+\tau)\Big) \\
={}& d\Big(\int_{-\infty}^{0} G_t(\tau+t,\varphi)A(\varphi_{\tau+t}(\varphi), u(\tau+t,\varphi_{\tau+t}(\varphi)),\varepsilon)\, d\tau \\
&+ \int_{-\infty}^{0} G_t(\tau+t,\varphi)Q(\varphi_{\tau+t}(\varphi), u(\tau+t,\varphi_{\tau+t}(\varphi)),\varepsilon)\, dW(t+\tau)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= d\Big(\int_{-\infty}^{t} G_t(\tau,\varphi)A(\varphi_\tau(\varphi),u(\tau,\varphi_\tau(\varphi)),\varepsilon)\,d\tau \\
&\quad + \int_{-\infty}^{t} G_t(\tau,\varphi)Q(\varphi_\tau(\varphi),u(\tau,\varphi_\tau(\varphi)),\varepsilon)\,dW(\tau)\Big) \\
&= (P(\varphi_t(\varphi))\int_{-\infty}^{t} G_t(\tau,\varphi)A(\varphi_\tau(\varphi),u(\tau,\varphi_\tau(\varphi)),\varepsilon)\,d\tau \\
&\quad + A(\varphi_t(\varphi),u(t,\varphi_t(\varphi)),\varepsilon)dt + (P(\varphi_t(\varphi))\int_{-\infty}^{t} G_t(\tau,\varphi) \\
&\quad \times Q(\varphi_\tau(\varphi),u(\tau,\varphi_\tau(\varphi)),\varepsilon)\,dW(\tau)dt + Q(\varphi_t(\varphi),u(t,\varphi_t(\varphi)),\varepsilon)dW(t) \\
&= P(\varphi_t(\varphi))u(t,\varphi_t(\varphi))dt + A(\varphi_t(\varphi),u(t,\varphi_t(\varphi)),\varepsilon)dt \\
&\quad + Q(\varphi_t(\varphi),u(t,\varphi_t(\varphi)),\varepsilon)dW(t),
\end{aligned}
$$

which proves that the pair $(\varphi_t(\varphi), u(t,\varphi_t(\varphi)))$ is a solution of system (4.37). This proves that the random function $u(t,\varphi)$ defines an invariant torus for system (4.37). □

4.5 An ergodic theorem for a class of stochastic systems having a toroidal manifold

Consider a system of stochastic differential Ito equations,

$$d\varphi = a(\varphi)dt,\ dx = (P(\varphi)x + A(\varphi,x))dt + \sum_{i=1}^{r} b_i(\varphi,x)dW_i(t)\,, \tag{4.45}$$

where $t \geq 0$, $x \in \mathbf{R}^n$, $\varphi = (\varphi_1,\dots\varphi_m) \in \mathbf{R}^m$, the functions $a(\varphi)$, $P(\varphi)$, $A(\varphi,x)$, $b_i(\varphi,x)$ are jointly continuous, 2π-periodic in φ_i $(i=\overline{1,m})$, the function $a(\varphi)$ is Lipschitz continuous in φ, and the functions A, b_i are Lipschitz continuous in $x \in \mathbf{R}^n$ with a constant L, $W_i(t)$, $i=\overline{1,m}$, are jointly independent Wiener processes.

Let

$$A(\varphi,0) = b_i(\varphi,0) = 0, \qquad i=\overline{1,m}\,. \tag{4.46}$$

It follows from condition (4.46) that system (4.45) has the invariant torus

$$x = 0, \qquad \varphi \in \Im_m\,. \tag{4.47}$$

Together with system (4.45), let us consider the deterministic system

$$\frac{d\varphi}{dt} = a(\varphi), \qquad \frac{dx}{dt} = P(\varphi)x\,. \tag{4.48}$$

Denote by $\Phi_\tau^t(\varphi)$ a matriciant of the system

$$\frac{dx}{dt} = P(\varphi_t(\varphi))x\,, \tag{4.49}$$

By [139, p. 121], we have the identity

$$\Phi_\tau^t(\varphi_\theta(\varphi)) = \Phi_{\tau+\theta}^{t+\theta}(\varphi).$$

Let $\Phi_0^t(\varphi)$ satisfy the condition

$$||\Phi_0^t(\varphi)|| \leq K \exp\{-\gamma t\} \tag{4.50}$$

for $t \geq 0$, $\varphi \in \Im_m$, and some positive constants K and γ.

Denote by $x(t, \varphi, x_0) = \Phi_0^t(\varphi)x_0$ the general solution of system (4.49). We have

$$\begin{aligned} |x(t,\varphi,x_0)| &= |\Phi_\tau^t(\varphi)\Phi_0^\tau(\varphi)x_0| = |\Phi_\tau^{t-\tau+\tau}(\varphi)x(\tau,\varphi,x_0)| \\ &\leq ||\Phi_0^{t-\tau}(\varphi_\tau(\varphi))|||x(\tau,\varphi,x_0)| \leq K\exp\{-\gamma(t-\tau)\}|x(\tau,\varphi,x_0)| \end{aligned}$$

for all $t \geq \tau \geq 0$ and arbitrary $\varphi \in \Im_m$, which gives the following estimate for the matriciant $\Phi_\tau^t(\varphi)$:

$$||\Phi_\tau^t(\varphi)|| \leq K \exp\{-\gamma(t-\tau)\}\,. \tag{4.51}$$

We will use it in the sequel.

Without loss of generality, to simplify the calculations, we assume that system (4.45) has only one scalar-valued Wiener process, and it has the form

$$d\varphi = a(\varphi)dt, \qquad dx = (P(\varphi)x + A(\varphi,x))dt + B(\varphi,x)dW(t)\,. \tag{4.52}$$

The following theorem clarifies the connection between stability of systems (4.49) and (4.52).

Theorem 4.8. *If the matriciant of system* (4.49) *satisfies estimate* (4.50) *and the constant L satisfies the estimate*

$$L < \frac{\gamma}{K(1+\gamma)^{\frac{1}{2}}}\,,$$

then a solution $x_t(\varphi, x_0)$ of system

$$dx = (P(\varphi_t(\varphi))x + A(\varphi_t(\varphi),x))dt + B(\varphi_t(\varphi),x)dW(t) \tag{4.53}$$

is totally mean square exponentially stable.

Proof. Let us show that $x_t(\varphi, x_0)$ can be represented as

$$x_t(\varphi, x_0) = \Phi_0^t(\varphi)x_0 + \int_0^t \Phi_\tau^t(\varphi)A(\varphi_\tau(\varphi), x_\tau(\varphi, x_0))\,d\tau + \int_0^t \Phi_\tau^t(\varphi)B(\varphi_\tau(\varphi), x_\tau(\varphi, x_0))\,dW(\tau)\,. \tag{4.54}$$

Indeed, it follows from [186, p. 264] that the random process

$$\eta(t) = \int_0^t \Phi_\tau^t(\varphi)B(\varphi_\tau(\varphi), x_\tau(\varphi, x_0))\,dW(\tau)$$

admits the stochastic differential

$$d\eta(t) = \left(\int_0^t \frac{\partial}{\partial t}\Phi_\tau^t(\varphi)B(\varphi_\tau(\varphi), x_\tau(\varphi, x_0)\right)dW(\tau))dt + \Phi_t^t(\varphi)B(\varphi_t(\varphi), x_\tau(\varphi, x_0))dW(t)\,. \tag{4.55}$$

So, using properties of the fundamental matrix and taking stochastic differential in (4.54) we get

$$\begin{aligned} dx_t(\varphi, x_0) = {} & P(\varphi_t(\varphi))\Phi_0^t(\varphi)x_0dt + P(\varphi_t(\varphi))\int_0^t \Phi_\tau^t(\varphi)A(\varphi_\tau(\varphi), x_\tau(\varphi, x_0))\,d\tau dt \\ & + P(\varphi_t(\varphi))\int_0^t \Phi_\tau^t(\varphi)B(\varphi_\tau(\varphi), x_\tau(\varphi, x_0))\,dW(\tau)dt \\ & + A(\varphi_t(\varphi), x_t(\varphi, x_0))dt + B(\varphi_t(\varphi), x_t(\varphi, x_0))dW(t)\,, \end{aligned}$$

which gives representation (4.54).

Let us estimate $\mathbf{E}|x_t(\varphi, x_0)|^2$. For this, firstly note that the conditions A and B give the estimates

$$|A(\varphi, x)| \le L|x|, \qquad |B(\varphi, x)| \le L|x|$$

for $x \in \mathbf{R}^n$, $\varphi \in \Im_m$, and L the Lipschitz constant.

Then, using (4.54) we have

$$\begin{aligned} \mathbf{E}|x_t(\varphi, x_0)|^2 \le {} & 3(||\Phi_0^t(\varphi)||^2|x_0|^2 + \mathbf{E}\left(\left|\int_0^t \Phi_\tau^t(\varphi)A(\varphi_\tau(\varphi), x_\tau(\varphi, x_0))\,d\tau\right|\right)^2 \\ & + \int_0^t ||\Phi_\tau^t(\varphi)||^2\mathbf{E}|B(\varphi_\tau(\varphi), x_\tau(\varphi, x_0))|^2\,dt) \\ & \le 3(K^2\exp\{-2\gamma t\}|x_0|^2 \end{aligned}$$

$$+\,\mathbf{E}\Big(\int_0^t K\exp\{-\gamma(t-\tau)\}|A(\varphi_\tau(\varphi),x_\tau(\varphi,x_0))|\,dt\Big)^2$$
$$+\int_0^t K^2\exp\{-2\gamma(t-\tau)\}L^2\mathbf{E}|x_\tau(\varphi,x_0)|^2\,d\tau)\,. \tag{4.56}$$

Applying the Cauchy–Bunyakovskii inequality to the second term in (4.56) we get the estimate

$$\begin{aligned}
&\mathbf{E}\Big(\int_0^t K\exp\{-\gamma(t-\tau)\}|A(\varphi_\tau(\varphi),x_\tau(\varphi,x_0))|\,dt\Big)^2\\
&\quad\le \mathbf{E}\Big(\int_0^t K\exp\left\{-\frac{1}{2}\gamma(t-\tau)\right\}L\exp\left\{-\frac{1}{2}\gamma(t-\tau)\right\}|x_\tau(\varphi,x_0)|\,d\tau\Big)^2\\
&\quad\le K^2L^2\int_0^t \exp\{-\gamma(t-\tau)\}\,d\tau\int_0^t \exp\{-\gamma(t-\tau)\}\mathbf{E}|x_\tau(\varphi,x_0)|^2\,d\tau\,,
\end{aligned} \tag{4.57}$$

and substituting representation (4.57) into (4.56) we get the estimate

$$\mathbf{E}|x_t(\varphi,x_0)|^2\le 3(K^2\exp\{-\gamma t\}|x_0|^2+\frac{K^2L^2}{\gamma}\int_0^t\exp\{-\gamma(t-\tau)\}\mathbf{E}|x_\tau(\varphi,x_0)|^2\,d\tau$$
$$+\,K^2L^2\int_0^t\exp\{-\gamma(t-\tau)\}\mathbf{E}|x_\tau(\varphi,x_0)|^2\,d\tau).$$

This shows that

$$\exp\{\gamma t\}\mathbf{E}|x_t(\varphi,x_0)|^2\le 3K^2|x_0|^2+\Big(\frac{K^2L^2}{\gamma}+K^2L^2\Big)\int_0^t\exp\{\gamma\tau\}\mathbf{E}|x_\tau(\varphi,x_0)|^2\,d\tau\,,$$

or

$$\exp\{\gamma t\}\mathbf{E}|x_t(\varphi,x_0)|^2\le 3K^2|x_0|^2\exp\left\{\Big(\frac{K^2L^2}{\gamma}+K^2L^2\Big)t\right\}.$$

Hence, finally,

$$\mathbf{E}|x_t(\varphi,x_0)|^2\le C\exp\left\{\Big(\frac{K^2L^2}{\gamma}+K^2L^2-\gamma\Big)t\right\}|x_0|^2\,,$$

which, using the conditions of the theorem, gives the inequality

$$M|x_t(\varphi,x_0)|^2\le C\exp\{-\gamma_1 t\}|x_0|^2\,, \tag{4.58}$$

where

$$\gamma_1=\frac{K^2L^2}{\gamma}+K^2L^2-\gamma\,.$$

Similar calculations for the segment $[\tau,\ t]$ for $0 \le \tau \le t$ and a use of estimate (4.51) prove for an arbitrary solution $x_t(\varphi, x_0)$ that

$$\mathbf{E}|x_t(\varphi, x_0)|^2 \le C \exp\{-\gamma_1(t-\tau)\}\mathbf{E}|x_\tau(\varphi, x_0)|^2\,, \tag{4.59}$$

which proves the theorem. □

This theorem shows that if estimate (4.50) holds, then the invariant torus $x = 0$, $\varphi \in \Im_m$, for system (4.52) is mean square totally exponentially stable.

If conditions of Theorem 4.8 are satisfied, it then follows from [81] that solutions of system (4.53) are totally exponentially stable with probability 1. More precisely, the following is true.

Corollary 4.2. *If conditions of Theorem 4.8 are satisfied, then solutions $x_t(\varphi, x_0)$ satisfy the estimate*

$$|x_t(\varphi, x_0)| \le C \exp\{-\alpha t\}|x_0|\,. \tag{4.60}$$

with probability 1 *starting with some finite time, random in general, with a positive constant $\alpha = \alpha(\gamma_1)$.*

This corollary asserts that the invariant torus $x = 0$, $\varphi \in \Im_m$, for system (4.52) is not only mean square stable but that it attracts all other solutions exponentially with probability 1, which shows that the torus $x = 0$, $\varphi \in \Im_m$, for system (4.50) is exponentially stable with probability 1.

Assume now that the winding of the torus $\Im_m$ is quasiperiodic, that is

$$d\varphi = \nu dt\,,$$

where $\nu = (\nu_1, \dots \nu_m)$ is a frequency basis for quasiperiodic solutions.

Under these conditions, the behavior of solutions of systems (4.45) and (4.52) are ergodic, which states the following theorem.

Theorem 4.9. *If the conditions of Theorem 4.8 are satisfied and the winding is quasiperiodic, then, for an arbitrary function $F(x, \varphi)$, which is continuous for $x \in \mathbf{R}^n$, $\varphi \in \Im_m$, and periodic in φ_i with period 2π, and an arbitrary solution $(x_t(\varphi, x),\ \varphi_t(\varphi))$ of system* (4.52), *the following limit relation holds with probability* 1*:*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T F(x_t(\varphi,x_0),\varphi_t(\varphi))\,dt = F_0$$
$$= (2\pi)^{-m}\int_0^{2\pi}\dots\int_0^{2\pi} F(0,\varphi)\,d\varphi_1\dots d\varphi_m\,. \tag{4.61}$$

Proof. Take an arbitrary solution of system (4.52) and consider its trajectory $(x_t(\varphi,x_0,\omega),\varphi_t(\varphi))$. It follows from the Corollary of Theorem 4.8 that there exists $T_0(\omega)$ such that $x_t(\varphi,x_0,\omega)$ satisfies estimate (4.60) for $t\ge T_0(\omega)$ and the inequality $|x_t(\varphi,x_0,\omega)|\le\delta$ with some fixed $\delta>0$ and almost all $\omega\in\Omega$. Approximate the function F with a polynomial in x for $|x|\le\delta$ so that

$$|F(x,\varphi)-P(x,\varphi,\varepsilon)|\le\varepsilon$$

in the domain $x:|x|\le\delta$, $\varphi\in\Im_m$, for fixed $\varepsilon>0$. This gives the estimate

$$\frac{1}{T}\left|\int_{T_0(\omega)}^T [F(x_t(\varphi,x_0,\omega),\varphi_t(\varphi)) - P(x_t(\varphi,x_0,\omega),\varphi_t(\varphi),\varepsilon)]\,dt\right|\le\varepsilon \tag{4.62}$$

for arbitrary $T\ge T_0(\omega)$.

However, since

$$\max_{|x|\le\delta,\varphi\in\Im_m}\left|\frac{\partial P(x,\varphi,\varepsilon)}{\partial x}\right| = L(\varepsilon)<\infty$$

by inequality (4.60), we have

$$\frac{1}{T}\int_{T_0(\omega)}^T [P(x_t(\varphi,x_0,\omega),\varphi_t(\varphi),\varepsilon)-P(0,\varphi_t(\varphi),\varepsilon)]\,dt$$
$$\le\frac{1}{T}\int_{T_0(\omega)}^T L(\varepsilon)|x_t(\varphi,x_0,\omega)|\,dt\le\frac{L(\varepsilon)}{T}\int_{T_0(\omega)}^T C\exp\{-\alpha t\}|x_0|\,dt\le\frac{L(\varepsilon)C|x_0|}{\alpha T}\,. \tag{4.63}$$

Approximate now the function $P(0,\varphi,\varepsilon)$ with a trigonometric polynomial $Q(\varphi,\varepsilon)$ in such a way that

$$|P(0,\varphi,\varepsilon)-Q(\varphi,\varepsilon)|\le\varepsilon$$

for arbitrary $\varphi\in\Im_m$. We get that

$$\frac{1}{T}\int_{T_0(\omega)}^T |P(0,\varphi_t(\varphi),\varepsilon)-Q(\varphi_t(\varphi),\varepsilon)|\,dt\le\varepsilon\,. \tag{4.64}$$

However,
$$Q(\varphi,\varepsilon)=\sum_{||k||\le N}Q_k(\varepsilon)\exp\{i(k,\varphi)\}\,,$$
where $N=N(\varepsilon)$ is a sufficiently large positive number, $Q_k(\varepsilon)$ are Fourier coefficients of the function $Q(\varphi,\varepsilon)$. Since the winding of the torus is quasiperiodic, $\varphi_t(\varphi)=\nu t+\varphi$ and, hence,
$$\frac{1}{T}\int_0^T Q(\nu t+\varphi,\varepsilon)\,dt=Q_0(\varepsilon)+\frac{1}{T}\int_0^T\sum_{1\le||k||\le N}Q_k(\varepsilon)\exp\{i(k,\nu)t\}\exp\{i(k,\varphi)\}\,dt,$$
where
$$Q_0(\varepsilon)=\frac{1}{(2\pi)^m}\int_0^{2\pi}\dots\int_0^{2\pi}Q(\varphi,\varepsilon)\,d\varphi_1\dots d\varphi_m$$
is the mean value of $Q(\varphi,\varepsilon)$.

It is clear that
$$\begin{aligned}\frac{1}{T}&\left|\int_0^T\sum_{1\le||k||\le N}Q_k(\varepsilon)\exp\{i(k,\nu)t\}\exp\{i(k,\varphi)\}\,dt\right|\\&\le\frac{1}{T}R(\varepsilon)\sum_{1\le||k||\le N}\left|\int_0^T\exp\{i(k,\nu)t\}\,dt\right|\\&=\frac{1}{T}R(\varepsilon)\sum_{1\le||k||\le N}\left|\frac{\exp\{i(k,\nu)T\}-1}{i(k,\nu)}\right|,\end{aligned}$$
which gives
$$\left|\frac{1}{T}\int_0^T Q(\nu t+\varphi,\varepsilon)\,dt-Q_0(\varepsilon)\right|\le\frac{1}{T}R_1(\varepsilon)\,. \tag{4.65}$$
We now have
$$|F_0-Q_0|\le|F_0-P_0|+|P_0-Q_0|\le 2\varepsilon\,, \tag{4.66}$$
where P_0 is the mean value of the polynomial $P(0,\varphi)$. However,
$$\begin{aligned}\left|\frac{1}{T}\int_0^T F(x_t(\varphi,x_0,\omega),\varphi_t(\varphi))\,dt-F_0\right|&\le\frac{1}{T}\left|\int_0^{T_0(\omega)}F(x_t(\varphi,x_0,\omega),\varphi_t(\varphi)\,dt\right|\\&+\left|\frac{1}{T}\int_{T_0(\omega)}^T F(x_t(\varphi,x_0,\omega),\varphi_t(\varphi)\,dt-F_0\right|.\end{aligned} \tag{4.67}$$
The first term in (4.67) tends to zero as $T\to\infty$, hence, for sufficiently large T, we have
$$\frac{1}{T}\left|\int_0^{T_0(\omega)}F(x_t(\varphi,x_0,\omega),\varphi_t(\varphi))\,dt\right|\le\varepsilon\,. \tag{4.68}$$

Inequalities (4.62)–(4.66) give the following for the second term in (4.67):

$$
\begin{aligned}
&\left|\frac{1}{T}\int_{T_0(\omega)}^{T} F(x_t(\varphi, x_0, \omega), \varphi_t(\varphi))\,dt - F_0\right| \\
&\quad\le \left|\frac{1}{T}\int_{T_0(\omega)}^{T} [F(x_t(\varphi, x_0, \omega), \varphi_t(\varphi)) - P(x_t(\varphi, x_0, \omega), \varphi_t(\varphi), \varepsilon)]\,dt\right| \\
&\qquad + \frac{1}{T}\left|\int_{T_0(\omega)}^{T} [P(x_t(\varphi, x_0, \omega), \varphi_t(\varphi), \varepsilon) - P(0, \varphi_t(\varphi), \varepsilon)]\,dt\right| \\
&\qquad + \frac{1}{T}\left|\int_{T_0(\omega)}^{T} [P(0, \varphi_t(\varphi), \varepsilon) - Q(\varphi_t(\varphi), \varepsilon)]\,dt\right| \\
&\qquad + \left|\frac{1}{T}\int_0^{T} Q(\nu t + \varphi, \varepsilon)\,dt - Q_0(\varepsilon)\right| + \frac{1}{T}\left|\int_0^{T_0(\omega)} Q(\nu t + \varphi, \varepsilon)\,dt\right| \\
&\qquad + |Q_0(\varepsilon) - F_0| \le \varepsilon + \frac{L(\varepsilon)C|x_0|}{\alpha T} + \varepsilon + \frac{1}{T}R_1(\varepsilon) \\
&\qquad + 2\varepsilon + \frac{1}{T}\left|\int_0^{T_0(\omega)} Q(\nu t + \varphi, \varepsilon)\,dt\right|.
\end{aligned}
\tag{4.69}
$$

Choose T so large that (4.68) is satisfied together with the inequality

$$
\frac{L(\varepsilon)C|x_0|}{\alpha T} + \frac{1}{T}R_1(\varepsilon) + \frac{1}{T}\left|\int_0^{T_0(\omega)} Q(\nu t + \varphi, \varepsilon)\,dt\right| \le \varepsilon.
$$

For such a choice of $T(\omega)$, we finally get that

$$
\left|\frac{1}{T}\int_0^{T} F(x_t(\varphi, x_0, \omega), \varphi_t(\varphi))\,dt - F_0\right| \le 5\varepsilon,
$$

which finishes the proof. □

The result obtained in Theorem 4.9 can be reformulated in terms of an ergodic measure.

Indeed, consider the measure $\mu(d\varphi) = d\varphi_1 \dots d\varphi_m\ \varphi_i \in [0,\ 2\pi],\ i = \overline{1, m}$, on the torus $\Im_m$. Construct the probability measure $\sigma(A) = \frac{\mu(A)}{(2\pi)^m}$, where A is a Borel subset of the torus $\Im_m$. The measure can be considered as a measure on the Cartesian product $\mathbf{R}^n \times \Im_m$ with the support on the torus $\Im_m$. Then, for an arbitrary function $F(x, \varphi)$, continuous on $\mathbf{R}^n \times \Im_m$ and periodic in φ, we have

$$
\int_{\mathbf{R}^n \times \Im_m} F(x, \varphi)\,\sigma(dx d\varphi) = \frac{1}{(2\pi)^m}\int_{\Im_m} F(0, \varphi)\,d\varphi_1 \dots d\varphi_m.
$$

This formula shows that the identity (4.61) in Theorem 4.9 can be written as

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(x_t(\varphi, x_0,), \varphi_t(\varphi_0))\, dt = \int_{\mathbf{R}^n \times \Im_m} F(x, \varphi)\, \sigma(dxd\varphi)$$

with probability 1, where σ is an ergodic measure with support in $\Im_m$.

There is a large number of works dealing with ergodic properties of solutions of stochastic Ito equations, see e.g. the monographs [70] and [156]. However, an essential assumption there is that the diffusion matrix is nondegenerate in a bounded domain and that the mean return time for a solution into the domain is finite (condition B is used in [70, p. 153]) or else that the solution does not have an invariant set distinct from the whole space (irreducibility of the process is used in [156, p. 66]).

In the case we have considered, these conditions are obviously violated, since system (4.45) has the invariant manifold $x = 0$, $\varphi \in \Im_m$, where the diffusion is degenerate (the system restricted to the manifold becomes deterministic).

At the end of this section, let us remark that oscillating systems that undergo an action by random factors were studied by many authors who used various approaches. An important class of mechanical systems, the Hamiltonian systems, were studied in [48] by using a stochastic version of the action functional. One can find there an extensive bibliography on the subject.

By considering a particular form of the flow on the m-dimensional torus $\Im_m$ (the first equation in (4.45)), one can obtain more precise results on the behaviour of solutions of such oscillating systems. By combining the averaging method and the method of martingale approximations, a number of interesting results was obtained in [157] in regard to asymptotic behavior of solutions of a linear oscillating system that is influenced by small random effects,

$$\frac{d}{dt} x_\varepsilon(t) = v_\varepsilon(t),$$

$$\frac{d}{dt} v_\varepsilon(t) = -\Lambda x_\varepsilon(t) + F(\varepsilon, t, x_\varepsilon, v_\varepsilon, \omega),$$

where Λ is a nonnegative definite matrix, and the random perturbations are either fast Markov of the form

$$F(\varepsilon, t, x_\varepsilon, v_\varepsilon, \omega) = f\left(x, v, y\left(\frac{t}{\varepsilon}\right)\right),$$

($y(t, \omega)$ is a homogeneous Markov process), or small white noise type perturbations,

$$F(\varepsilon, t, x_\varepsilon, v_\varepsilon, \omega) = \sqrt{\varepsilon} F(x, v) \frac{d}{dt} w(t),$$

where $w(t)$ is an m-dimensional Wiener process.

4.6 Comments and References

Section 4.1. Oscillation theory has its origins in celestial mechanics. Mathematical models for studying processes were the most simple differential equations that would reduce to linear equations, see, e.g. Krylov [85], Lord Rayleigh [130]. Later, the needs of electrical and radio engineering have called for a development of a perturbation theory for weakly nonlinear systems. A fundamental contribution was made there by M. M. Krylov and M. M. Bogolyubov [19, 21, 86]. The most important objects in the theory were periodic solutions of weakly nonlinear differential systems, as well as more complex objects that are invariant sets for such systems.

Starting in sixties in the last century, there was a drastic turn in the oscillation theory towards a study of oscillation processes that repeat "almost exactly" on "almost the same" time interval. Such processes are called multi-frequence oscillations and are described in terms of quasiperiodic functions. The most important achievement in this direction is a creation of the KAM-theory (Kolmogorov [75], Arnold [12], Moser [116]), which is a theory of quasiperiodic solutions of "almost integrable" Hamiltonian systems. About stochastic systems at this direction we mention the works Johansson, Kopidakis and Aubry [63], Appleby, Kelly [6] and Gitterman [55].

However, quasiperiodic oscillations make an object rather difficult to study and unstable under perturbations. This fact has initiated a search for an object in the theory that would be coarser than a quasiperiodic solution. It turned out that a "carrier" of quasiperiodic solutions is a torus. It is this torus that is "swept" by the quasiperiodic function. This makes it of current interest to study conditions for the oscillating system to have an invariant torus. Under fairly general conditions, the torus is a coarse object that is not usually destroyed under small perturbations, but is only deformed. When studying invariant tori, a fruitful notion is that of Green's function for the invariant tori. The use of it allows to give an integral representation for the invariant torus. This notion gave a new stimulus to the development of various aspects of the theory and led to new results exposed in the monograph of Samoilenko [139] in a detailed way.

The oscillation theory in systems with random perturbations has been very little studied at this time. In this connection, one should mention systematic studies related to the averaging method for systems described by second order equations with randomly perturbed coefficients, see Kolomiets' [76, 77]. Certain questions in the study of oscillating systems influenced by random forces have already been studied in the cited monograph of Freidlin and Wentzell [48], dealing with Hamiltonian systems, in the monograph of Arnold [8], and in the monograph of Skorokhod [156]. Let us also mention the monograph of Stratonovich [177], where a large number of applied problems that have oscillations were considered. However, no studies of invariant tori appearing in such systems were carried out. The results given in this section were obtained in the works of Stanzhyts'kyj and Kopas' [173], Stanzhyts'kyj [160, 166].

Sections 4.2–4.4. Integral representations for invariant tori were obtained in the deterministic case by Samoilenko in [139]. For stochastic Ito systems, such questions have not been considered. The content of these sections is based on the work of Samoilenko and Stanzhyts'kyj [149], Stanzhyts'kyj [170], Samoilenko, Stanzhyts'kyj and Ateiwi [150].

Section 4.5. Ergodic behavior of solutions of stochastic systems were studied in many works, see e.g. the monographs of Khas'mins'kyi [70], Skorokhod [156], Arnold [8], and his paper [9]. The results of this section appeared in the work of Stanzhyts'kyj [176] that is a generalization, to the stochastic case, of Samoilenko's result from the monograph of Samoilenko and Petrishin [145].

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Chapter 5

The averaging method for equations with random perturbations

In this chapter, we substantiate the averaging method for impulsive systems with random perturbations that are not of Markov type, study the asymptotics of normalized deviations between the exact and the averaged solutions. We also prove here, for differential systems with regular random perturbations, an analogue of the second theorem of Bogolyubov. For stochastic Ito systems, we obtain versions of the first and the second theorems of Bogolyubov, where the convergence is regarded in the mean square sense, and consider some other questions.

In Section 5.1, we substantiate the Bogolyubov averaging method applied, on bounded time intervals, to differential systems with a random impulsive effect and a small parameter. We prove that solutions of the initial system and the averaged system, which is deterministic in this case, are close in the mean square sense.

Section 5.2 studies the asymptotic behaviour of the normalized deviations between the exact and the averaged motions of impulsive systems. We show that such deviations weakly converge to solutions of a linear stochastic Ito system, an exact form of which is given. Essentially, it is a system in variations for derivatives of solutions of the exact system with respect to the parameter.

Section 5.3 deals with applications. Here, we apply the preceding results to study small nonlinear oscillations.

Section 5.4 substantiates the averaging method for systems with impulsive effects occurring at random times.

Section 5.5 generalized the second Bogolyubov theorem to differential systems with a random right-hand side.

In Sections 5.6–5.7, for stochastic Ito systems, we prove an analogue of the first Bogolyubov theorem and the Banfi-Filatov averaging theorem for both bounded and unbounded time intervals. As opposed to the results known before, we prove here that solutions of the exact and the averaged systems are mean square close.

Section 5.8 deals with a construction of a two-sided mean square bounded solution of a stochastic system. This construction is carried out by using the averaging method developed in the preceding sections. We show here that an asymptotically stable equilibrium of the averaged system generates, in its neighborhood, a two-sided solution, which is mean square bounded on the axis, of the initial stochastic Ito system.

5.1 A substantiation of the averaging method for systems with impulsive effect

In this section, we propose and substantiate an averaging scheme for differential systems with random impulsive effects occurring at fixed times.

Let (Ω, F, P) be a probability space. Consider a differential system with a random right-hand side and a random impulsive effects that occur at fixed times,

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon X(t,x,w), \qquad t \neq t_i \\ \Delta x|_{t=t_i} &= x(t_i+0,w) - x(t_i-0,w) = \varepsilon I_i(x,w)\,, \end{aligned} \tag{5.1}$$

where $i = 1, 2, .., \varepsilon$ is a small positive parameter. We assume that system (5.1) satisfies the following conditions.

1) For every $x \in \mathbf{R}^n$ and $i \in \mathbf{N}$, $X(t,x,w)$ is a measurable random process, and $I_i(x,w)$ is a random variable, both defined on (Ω, F, P).

2) The functions $X(t,x,w)$ and $I_i(x,w)$ are continuous in $x \in \mathbf{R}^n$ with probability 1.

3) There exist $C > 0$ and $K > 0$ such that

$$\mathbf{E}|X(t,x,w)| + \mathbf{E}|I_i(x,w)| \leq C \qquad \forall t \geq 0, \;\; \forall x \in \mathbf{R}^n\,,$$

and

$$|X(t,x^{'},w)-X(t,x^{''},w)|+|I_i(x^{'},w)-I_i(x^{''},w)|\le K|x'-x''|\,,$$

with probability 1 for arbitrary $t \ge 0$, and $x^{'}, x^{''} \in \mathbf{R}^n$.

We will assume that all solutions of system (5.1) can be unboundedly continued to the right. This is the case, for example, if the function $X(t,x,w)$ has linear growth with respect to x as $|x| \to \infty$.

We give an averaging scheme and substantiate it for systems of the form (5.1) over a bounded time interval of length order $O(\frac{1}{\varepsilon})$.

The times of the impulsive effects are assumed to satisfy the following conditions:

$$\sum_{0<t_i<T} 1 \le CT, \qquad C>0\,. \tag{5.2}$$

Theorem 5.1. *Let system* (5.1) *satisfy the above conditions 1), 2), 3) and, moreover, there exist functions* $X_0(x)$ *and* $I_0(x)$ *such that the law of large numbers in the following form holds uniformly in* $x \in R^n$*:*

$$\begin{aligned}\mathbf{E}\left|\frac{1}{T}\int_0^T X(s,x,w)ds - X_0(x)\right| &\to 0 \quad as \quad T\to\infty\,,\\ \mathbf{E}\left|\frac{1}{T}\sum_{0<t_i<T} I_i(x,w) - I_0(x)\right| &\to 0 \quad as \quad T\to\infty\,.\end{aligned} \tag{5.3}$$

Let also condition (5.2) *hold.*

If $x(t,x_0)$, $x(0,x_0)=x_0$, *is a solution of system* (5.1) *and* $\bar{x}=\bar{x}(t,x_0)$, $\bar{x}(0,x_0)=x_0$, *is a solution of the averaged system*

$$\frac{d\bar{x}}{dt} = \varepsilon[X_0(\bar{x})+I_0(\bar{x})]\,, \tag{5.4}$$

then for arbitrary $L>0$ *and arbitrary small* $\eta>0$ *there exists* $\varepsilon_0>0$ *such that the following estimate holds for all* $\varepsilon<\varepsilon_0$ *and* $t\in[0,\frac{L}{\varepsilon}]$*:*

$$\mathbf{E}|x(t,x_0)-\bar{x}(t,x_0)| < \eta\,. \tag{5.5}$$

Proof. First of all note that conditions of the theorem imply existence and uniqueness of a solution $\bar{x}=\bar{x}(t,x_0)$ for arbitrary $t\ge 0$. Indeed, it follows from (5.3), by uniform convergence, that $X_0(x), I_0(x)$ are continuous, bounded, and Lipschitz continuous for arbitrary $x\in\mathbf{R}^n$. Moreover, it follows from (5.3) that there is a monotone decreasing function $\varphi(t)$, which converges to zero as $t \to \infty$, such that

$$\begin{aligned} \mathbf{E}\left| \int_0^T [X(s,x,w) - X_0(x)]ds \right| &\le \frac{\varphi(T)}{2} T\,, \\ \mathbf{E}\left| \sum_{0<t_i<T} I_i(x,w) - I_0(x)T \right| &\le \frac{\varphi(T)}{2} T\,. \end{aligned} \tag{5.6}$$

Write systems (5.1) and (5.4) in the integral-summation and the integral forms, respectively,

$$x(t,x_0,w) = x_0 + \varepsilon \int_0^t X(s,x(s),w)ds + \varepsilon \sum_{0<t_i<t} I_i(x(t_i),w)\,,$$

$$\bar{x}(t,x_0) = x_0 + \varepsilon \int_0^t [X_0(\bar{x}(s)) + I_0(\bar{x}(s))]ds.$$

We will drop the arguments x_0 and w in x and $\bar{x}$, correspondingly. Then

$$\begin{aligned} |x(t) - \bar{x}(t)| \le \Bigg| \varepsilon & \int_0^t [X(s,x(s),w) - X(s,\bar{x}(s),w)]ds \\ & + \varepsilon \sum_{0<t_i<t} [I_i(x(t_i),w) - I_i(\bar{x}(t_i),w)] \\ & + \varepsilon \int_0^t [X(s,\bar{x}(s),w) - X_0(\bar{x}(s))]ds \\ & + \varepsilon \sum_{0<t_i<t} I_i(\bar{x}(t_i),w) - \varepsilon \int_0^t I_0(\bar{x}(s))ds \Bigg| \le \varepsilon \int_0^t K|x(s) - \bar{x}(s)|ds \\ & + \varepsilon \sum_{0<t_i<t} K|x(t_i) - \bar{x}(t_i)| + \varepsilon \left| \int_0^t [X(s,\bar{x}(s),w) - X_0(\bar{x}(s))]ds \right| \\ & + \varepsilon \left| \sum_{0<t_i<t} I_i(\bar{x}(t_i),w) - \int_0^t I_0(\bar{x}(s))ds \right|. \end{aligned} \tag{5.7}$$

The integral term in (5.7) can be estimates essentially as in [66, p. 13].

This proves that for arbitrary $\eta > 0$ there exists $\varepsilon > 0$ such that

$$\mathbf{E}\varepsilon \left| \int_0^t [X(s, \bar{x}(s), w) - X_0(\bar{x}(s))]ds \right| \le \frac{\eta}{2} e^{-KL} e^{-CLK} . \tag{5.8}$$

Let us estimate the last term in (5.7). To this end, let us partition the segment $[0, \frac{L}{\varepsilon}]$ with points $\{\tau_k\}$ into n equal intervals. We get

$$\begin{aligned} &\varepsilon \left[\sum_{0<t_i<t} I_i(\bar{x}(t_i), w) - \int_0^t I_0(\bar{x}(s))ds \right] \\ &\quad \le \varepsilon \sum_{k=0}^{n-1} \left[\sum_{\tau_k \le t_i < \tau_{k+1}} I_i(\bar{x}(t_i), w) - \int_{\tau_k}^{\tau_{k+1}} I_0(\bar{x}(s))ds \right] . \end{aligned} \tag{5.9}$$

Now, estimate each term in the latter sum,

$$\begin{aligned} &\sum_{\tau_k \le t_i < \tau_{k+1}} (I_i(\bar{x}(t_i), w) - \int_{\tau_k}^{\tau_{k+1}} I_0(\bar{x}(s))ds) \\ &\quad = \sum_{\tau_k \le t_i < \tau_{k+1}} (I_i(\bar{x}(t_i), w) - I_0(\bar{x}(\tau_k)) \times (\tau_{k+1} - \tau_k) \\ &\qquad + I_0(\bar{x}(\tau_k))(\tau_{k+1} - \tau_k) - \int_{\tau_k}^{\tau_{k+1}} I_0(\bar{x}(s))ds) . \end{aligned} \tag{5.10}$$

However,

$$\begin{aligned} &\left| I_0(\bar{x}(\tau_k))(\tau_{k+1} - \tau_k) - \int_{\tau_k}^{\tau_{k+1}} I_0(\bar{x}(s))ds \right| \\ &\quad = \left| \int_{\tau_k}^{\tau_{k+1}} [I_0(\bar{x}(\tau_k)) - I_0(\bar{x}(s))]ds \right| \le \int_{\tau_k}^{\tau_{k+1}} K|\bar{x}(\tau_k)) - \bar{x}(s)|ds . \end{aligned} \tag{5.11}$$

Now we use

$$|\bar{x}(\tau_k)) - \bar{x}(s)| \le \varepsilon \int_{\tau_k}^{\tau_{k+1}} |I_0(\bar{x}(s)) + X_0(\bar{x}(s)|ds \le 2\varepsilon C \frac{L}{\varepsilon n} = 2\frac{CL}{n} . \tag{5.12}$$

By substituting (5.12) into (5.11), we get the estimate

$$\left| I_0(\bar{x}(\tau_k))(\tau_{k+1}-\tau_k) - \int\limits_{\tau_k}^{\tau_{k+1}} I_0(\bar{x}(s))ds \right| \le \frac{CL^2K}{n^2\varepsilon}\,. \tag{5.13}$$

It remains to find an estimate for the first term in (5.10). We have

$$\begin{aligned} &\sum_{\tau_k\le t_i<\tau_{k+1}} \left(I_i(\bar{x}(t_i),w) - I_0(\bar{x}(\tau_k))(\tau_{k+1}-\tau_k)\right) \\ &\quad= \sum_{\tau_k\le t_i<\tau_{k+1}} I_i(\bar{x}(t_i),w) - \sum_{\tau_k\le t_i<\tau_{k+1}} I_i(\bar{x}(\tau_k),w) \\ &\qquad+ \sum_{\tau_k\le t_i<\tau_{k+1}} I_i(\bar{x}(\tau_k),w) - I_0(\bar{x}(\tau_k))(\tau_{k+1}-\tau_k)\,. \end{aligned} \tag{5.14}$$

For the first term in (5.14), we have

$$\begin{aligned} &\sum_{\tau_k\le t_i<\tau_{k+1}} |I_i(\bar{x}(t_i),w) - I_i(\bar{x}(\tau_k),w)| \\ &\quad\le \sum_{\tau_k\le t_i<\tau_{k+1}} K|\bar{x}(t_i)-\bar{x}(\tau_k)| \le \sum_{\tau_k\le t_i<\tau_{k+1}} \frac{CKL}{n}\,, \end{aligned}$$

and, hence,

$$\begin{aligned} \varepsilon\sum_{k=0}^{n-1}\sum_{\tau_k\le t_i<\tau_{k+1}} |I_i(\bar{x}(t_i),w) - I_i(\bar{x}(\tau_k),w)| &\le \varepsilon\sum_{k=0}^{n-1}\sum_{\tau_k\le t_i<\tau_{k+1}} \frac{CKL}{n} \\ &\le \varepsilon\frac{CKL}{n}\frac{CL}{\varepsilon} = \frac{C^2L^2K}{n}\,. \end{aligned} \tag{5.15}$$

It follows from (5.10)–(5.13) and (5.9) that

$$\begin{aligned} &\varepsilon\left|\sum_{k=0}^{n-1}\left[\sum_{\tau_k\le t_i<\tau_{k+1}} I_i(\bar{x}(t_i),w) - \int\limits_{\tau_k}^{\tau_{k+1}} I_0(\bar{x}(s))ds\right]\right| \\ &\le \varepsilon\sum_{k=0}^{n-1}\left(\frac{CL^2K}{n^2\varepsilon} + \left|\sum_{\tau_k\le t_i<\tau_{k+1}} I_i(\bar{x}(t_i),w) - I_0(\bar{x}(\tau_k))(\tau_{k+1}-\tau_k)\right|\right) \\ &= \frac{CL^2K}{n} + \varepsilon\left|\sum_{k=0}^{n-1}\left(\sum_{\tau_k\le t_i<\tau_{k+1}} I_i(\bar{x}(t_i),w) - I_0(\bar{x}(\tau_k))(\tau_{k+1}-\tau_k)\right)\right|. \end{aligned} \tag{5.16}$$

Using (5.14) and (5.15) for estimating the second term in the last formula, we have

$$\varepsilon\left|\sum_{k=0}^{n-1}\left(\sum_{\tau_k\le t_i<\tau_{k+1}} I_i(\bar{x}(t_i),w)-I_0(\bar{x}(\tau_k))(\tau_{k+1}-\tau_k)\right)\right|\le\frac{C^2L^2K}{n}$$
$$+\ \varepsilon\left|\sum_{k=0}^{n-1}\sum_{\tau_k\le t_i<\tau_{k+1}} I_i(\bar{x}(\tau_k),w)-I_0(\bar{x}(\tau_k))(\tau_{k+1}-\tau_k)\right|.$$

Hence, by (5.16) and the above inequality, we have

$$\varepsilon\left|\sum_{\tau_k\le t_i<\tau_{k+1}} I_i(\bar{x}(t_i),w)-\int\limits_0^t I_0(\bar{x}(s))ds\right|\le\frac{2C^2L^2K}{n}$$
$$+\ \varepsilon\left|\sum_{k=0}^{n-1}\left(\sum_{\tau_k\le t_i<\tau_{k+1}} I_i(\bar{x}(\tau_k),w)-I_0(\bar{x}(\tau_k))(\tau_{k+1}-\tau_k)\right)\right|. \quad (5.17)$$

Let us now choose n such that

$$\frac{2C^2L^2K}{n}\le\frac{\eta}{4}e^{-KL}e^{-CLK}, \quad (5.18)$$

and fix it. Calculate the expectation of the second term in (5.17) using the second inequality in (5.6). We have

$$\varepsilon\mathbf{E}\left|\sum_{k=0}^{n-1}\left(\sum_{\tau_k\le t_i<\tau_{k+1}} I_i(\bar{x}(\tau_k),w)-I_0(\bar{x}(\tau_k))(\tau_{k+1}-\tau_k)\right)\right|$$
$$=\varepsilon\mathbf{E}\left|\sum_{k=0}^{n-1}\left(\sum_{0<t_i<\tau_{k+1}} I_i(\bar{x}(\tau_k),w)-I_0(\bar{x}(\tau_k))\tau_{k+1}\right.\right.$$
$$\left.\left.-\sum_{0<t_i<\tau_k} I_i(\bar{x}(\tau_k),w)+I_0(\bar{x}(\tau_k))\tau_k\right)\right|$$
$$\le\varepsilon\sum_{k=0}^{n-1}\mathbf{E}\left|\sum_{0<t_i<\tau_{k+1}} I_i(\bar{x}(\tau_k),w)-I_0(\bar{x}(\tau_k))\tau_{k+1}\right|$$
$$+\ \varepsilon\sum_{k=0}^{n-1}\mathbf{E}\left|\sum_{0<t_i<\tau_k} I_i(\bar{x}(\tau_k),w)-I_0(\bar{x}(\tau_k))\tau_k\right|$$

$$\leq \varepsilon \sum_{k=0}^{n-1} \frac{\tau_{k+1}\varphi(\tau_{k+1})}{2} + \varepsilon \sum_{k=0}^{n-1} \frac{\tau_k \varphi(\tau_k)}{2}$$

$$\leq \varepsilon \sum_{k=0}^{n-1} \frac{L}{\varepsilon}\varphi\left(\frac{L}{\varepsilon n}\right) = nL\varphi\left(\frac{L}{\varepsilon n}\right). \tag{5.19}$$

Choose ε to be so small that

$$nL\varphi\left(\frac{L}{\varepsilon n}\right) \leq \frac{\eta}{4} e^{-KL} e^{-CLK}. \tag{5.20}$$

Going back to (5.7), it follows from inequalities (5.8), (5.18), and (5.20) that

$$\mathbf{E}|x(t)-\bar{x}(t)| \leq \varepsilon \int_0^t K\mathbf{E}|x(s)-\bar{x}(s)|ds + \varepsilon \sum_{0<t_i<t} K\mathbf{E}|x(t_i)-\bar{x}(t_i)| + \eta e^{-KL} e^{-CLK}$$

that, together with a Gronwall-Bellman type inequality, gives

$$\mathbf{E}|x(t) - \bar{x}(t)| \leq \eta e^{-KL} e^{-CLK} e^{KL} (1+\varepsilon K)^{\frac{CL}{\varepsilon}} \leq \eta$$

for $t \in [0, \frac{L}{\varepsilon}]$. This proves the theorem. □

5.2 Asymptotics of normalized deviations of averaged solutions

Particularities of the probability case fully show in a study of the asymptotic behaviour of normalized deviations between the exact and the averaged solutions. A typical situation in the deterministic case is where the normalization is taken to be of order ε. It turns out that the stochastic case needs another normalization, that of order $\sqrt{\varepsilon}$. In this section, we study the asymptotics of normalized deviations.

Let us again consider system (5.1), assuming that conditions 1), 2), 3) in the preceding section are satisfied. Condition (5.2) is replaced with the following:

$$\sum_{t<t_i<T+t} 1 \leq CT, \qquad C > 0, \tag{5.21}$$

uniformly in $t > 0$, and the second condition in (5.3) with

$$\mathbf{E} \sup_{x \in \mathbf{R}^n} \left| \frac{1}{T} \sum_{t<t_i<T+t} I_i(x,w) - I_0(x) \right| \to 0 \quad \text{as} \quad T \to \infty \tag{5.22}$$

uniformly in $t > 0$.

Denote by $x(t, x_0, w)$, $x(0, x_0, w) = x_0$, and $\bar{x}(t, x_0)$ solutions of systems (5.1) and (5.4), respectively. To simplify the notations, x_0 and w will be dropped, and we simply write $x(t)$ and $\bar{x}(t)$, correspondingly.

Let us study the limit behaviour of the fluctuations

$$X^{(\varepsilon)}(t) = \frac{x(t) - \bar{x}(t)}{\sqrt{\varepsilon}} \tag{5.23}$$

of the solution of system (5.1) with respect to the solution of system (5.4) as $\varepsilon \to 0$.

The system

$$\frac{dy}{dt} = \varepsilon[X(t, y, w) + I_0(y)]\,, \tag{5.24}$$

where $I_0(y)$ is defined by relation (5.22), will be called "partially" averaged.

Lemma 5.1. *Let solutions $x(t)$ and $y(t)$, $x(0) = x_0$ and $y(0) = x_0$, of equations (5.1) and (5.24) exist for all $t \geq 0$, and conditions 1), 2), 3), (5.21) be satisfied, and the first condition in (5.3) and condition (5.22) hold. Then*

$$\mathbf{E}\left|X^{(\varepsilon)}\left(\frac{\tau}{\varepsilon}\right) - Y^{(\varepsilon)}\left(\frac{\tau}{\varepsilon}\right)\right| \to 0, \qquad \varepsilon \to 0\,,$$

for every $\tau \in [0,\ L]$, where

$$Y^{(\varepsilon)}\left(\frac{\tau}{\varepsilon}\right) = \frac{y(\frac{\tau}{\varepsilon}) - \bar{x}(\tau)}{\sqrt{\varepsilon}}\,. \tag{5.25}$$

Proof. Consider $\mathbf{E}|X^{(\varepsilon)}(t) - Y^{(\varepsilon)}(t)|$. Fix $T > 0$ and subdivide the segment $[0, \frac{L}{\varepsilon}]$ into intervals of length T. Then, for $t \in [kT, (k+1)T]$, we have

$$\begin{aligned}\mathbf{E}|X^{(\varepsilon)}(t) - Y^{(\varepsilon)}(t)| &= \frac{1}{\sqrt{\varepsilon}}\mathbf{E}|x(t) - y(t)| \\ &\leq \frac{1}{\sqrt{\varepsilon}}[\mathbf{E}|x(t) - x(kT)| + \mathbf{E}|y(t) - y(kT)| \\ &\quad + \mathbf{E}|x(kT) - y(kT)|]\,.\end{aligned} \tag{5.26}$$

Now,

$$\frac{1}{\sqrt{\varepsilon}}\mathbf{E}|x(t) - x(kT)| \leq \frac{1}{\sqrt{\varepsilon}}\mathbf{E}\Bigg[\varepsilon \int_{kT}^{(k+1)T} |X(t, x(t, x_0), w)|dt$$

$$+ \varepsilon \sum_{kT \le t_i < (k+1)T} |I_i(x(t, x_0), w)| \Bigg]$$

$$\le \frac{\varepsilon TC}{\sqrt{\varepsilon}} + \frac{\varepsilon Cd}{\sqrt{\varepsilon}} \le \sqrt{\varepsilon} C(T + d)\,, \tag{5.27}$$

since it follows from (5.21) that the number of points t_i on any interval of length T does not exceed the constant $d = CT$.

Now we have

$$\frac{1}{\sqrt{\varepsilon}} \mathbf{E}|y(t) - y(kT)| \le \frac{1}{\sqrt{\varepsilon}} \mathbf{E}\varepsilon \int_{kT}^{(k+1)T} |X(t, y(t), w) + I_0(y(t))| dt$$

$$\le 2\frac{\varepsilon TC}{\sqrt{\varepsilon}} = 2\sqrt{\varepsilon} TC\,. \tag{5.28}$$

Let us estimate $\frac{1}{\sqrt{\varepsilon}} \mathbf{E}|x(kT) - y(kT)|$. Similarly to [136], we have the identity

$$x(T) = x_0 + \varepsilon \int_0^T X(t, x_0, w) dt + \varepsilon \sum_{0 < t_i < T} I_i(x_0, w) + R_1(t, \varepsilon, T, d_1, w),$$

where d_1 is the number of impulses on $[0, T)$, and the estimate

$$\mathbf{E}|R_1(t, \varepsilon, T, d_1, w)| \le \varepsilon^2 M_1(T, d_1)\,,$$

as well as the identity

$$y(T) = x_0 + \varepsilon \int_0^T [X(t, x_0, w) + I_0(x_0)] dt + \bar{R}_1(t, \varepsilon, T)\,,$$

with the estimate

$$|\bar{R}_1(t, \varepsilon, T)| \le \varepsilon^2 \bar{M}_1(T).$$

Now we have

$$\mathbf{E}|x(T) - y(T)| \le \varepsilon \mathbf{E}\Big| \sum_{0 < t_i < T} I_i(x_0, w) - I_0(x_0)T \Big| + \varepsilon^2 \bar{\bar{M}}_1(T, d_1)\,.$$

It follows from (5.6) that

$$\mathbf{E}|x(T) - y(T)| \le \varepsilon \frac{\varphi(T)}{2} T + \varepsilon^2 \bar{\bar{M}}_1(T, d_1)\,. \tag{5.29}$$

Now,

$$\mathbf{E}|x(2T)-y(2T)| = \mathbf{E}\Bigg| x(T) + \varepsilon\int\limits_T^{2T} X(t,x(T),w)dt + \varepsilon\sum_{T<t_i<2T} I_i(x(T),w)$$

$$+ R_2(t,d_2,T,w) - y(T) - \varepsilon\int\limits_T^{2T}[X(t,y(T),w)+I_0(y(T))]dt + \bar{R}_2(t,\varepsilon,T)\Bigg|$$

$$\leq \mathbf{E}|x(T)-y(T)| + \varepsilon K\int\limits_T^{2T}\mathbf{E}|x(T)-y(T)|dt$$

$$+ \varepsilon\mathbf{E}\Bigg|\sum_{T<t_i<2T} I_i(x(T),w) - I_0(y(T))T\Bigg| + \varepsilon^2\bar{\bar{M}}_2$$

$$\leq \mathbf{E}|x(T)-y(T)| + \varepsilon KT\mathbf{E}|x(T)-y(T)| + \varepsilon^2\bar{M}_2$$

$$+ \varepsilon\mathbf{E}\Bigg|\sum_{T<t_i<2T} I_i(x(T),w) - I_0(y(T))T\Bigg| + \varepsilon T\mathbf{E}|I_0(y(T)) - I_0(x(T))|$$

$$\leq \mathbf{E}|x(T)-y(T)| + \varepsilon KT\mathbf{E}|x(T)-y(T)| + \varepsilon^2\bar{\bar{M}}_2 + \varepsilon\frac{\varphi(T)}{2}TK\mathbf{E}|x(T)-y(T)|. \tag{5.30}$$

By a similar procedure, we get the following estimate at the k-th step:

$$\mathbf{E}|x(kT)-y(kT)| \leq \varepsilon\frac{\varphi(T)}{2}T + \mathbf{E}|x((k-1)T) - y((k-1)T)|$$
$$+ 2\varepsilon KT\mathbf{E}|x((k-1)T) - y((k-1)T)| + \varepsilon^2\bar{\bar{M}}_k(T,d_{k-1})\,. \tag{5.31}$$

It follows from (5.29)–(5.31) that $\frac{1}{\sqrt{\varepsilon}}\mathbf{E}|x(kT)-y(kT)| \to 0$ as $\varepsilon\to 0$, so the expression (5.26) tends to zero as $\varepsilon \to 0$. □

It will be convenient for what follows to pass to a "slow" time in systems (5.1), (5.4), and (5.24) by the change of the variable $\tau = \varepsilon t$. Then (5.1) becomes

$$\begin{aligned} \frac{dx}{d\tau} &= X\left(\frac{\tau}{\varepsilon},x,w\right), \qquad \tau \neq \varepsilon t_i, \\ \Delta|_{\tau=\tau_i=\varepsilon t_i} &= \varepsilon I_i(x,w)\,, \end{aligned} \tag{5.32}$$

and (5.4) and (5.24) will take the corresponding form

$$\begin{aligned}\frac{d\bar{x}}{d\tau} &= X_0(\bar{x}) + I_0(\bar{x}),\\ \frac{dy}{d\tau} &= X\left(\frac{\tau}{\varepsilon}, y, w\right) + I_0(y)\,.\end{aligned} \tag{5.33}$$

We will consider these systems over the segment $[0, L]$. Let conditions of Lemma 5.1 hold. Moreover, assume that on the probability space (Ω, F, P) there is a family of σ-algebras F_s^t, $0 \le s \le t \le \infty$, of subsets of Ω satisfying the following conditions:

a) $F_s^t \subset F$, $\forall s, t$;

b) $F_s^t \subset F_{s_1}^{t_1}$ for $s_1 \le s \le t \le t_1$;

c) the function $X(t, x, w)$ is F_t^t-measurable, and $I_i(x, w)$ is $F_{t_i}^{t_i}$-measurable for every t, x, i;

d) the complete regularity condition holds, that is,

$$\beta(\tau) = \sup_{t\ge 0} \mathbf{E} var_{A\in F_{t+\tau}^{\infty}} \{P(A|_{F_0^t}) - P(A)\} \to 0$$

as $\tau \to \infty$.

Assume also the following.

4) $\mathbf{E}|X(t, x, w)|^{16} \le C$ $\forall t \ge 0$, $\forall x \in \mathbf{R}^n$, and $\mathbf{E}|I_i(x, w)|^{16} \le C$, $\forall i \in N$, $\forall x \in \mathbf{R}^n$;

5) $\mathbf{E}|\frac{\partial X(t,x,w)}{\partial x}|^{16} \le C$ and $|\frac{\partial I_0(x)}{\partial x}| \le C$ $\forall t \ge 0$, $\forall x \in \mathbf{R}^n$;

6) $\mathbf{E}|\frac{\partial^2 X(t,x,w)}{\partial x_i \partial x_j}|^2 + |\frac{\partial^2 I_0(x)}{\partial x_i \partial x_j})|^2 \le C$, $i, j = \overline{1, n}$.

Denote $f(t, x) = \mathbf{E}X(t, x, w)$, $A_{ij}(t, x) = \mathbf{E}\frac{\partial X_i}{\partial x_j} + \frac{\partial I_{0i}}{\partial x_j}$, $C_{ij}(t, s, x) = \mathbf{E}\{[X_i(t, x, w) - \mathbf{E}X_i(t, x, w)][X_j(s, x, w) - \mathbf{E}X_j(s, x, w)]\}$.

7) The following limits exist uniformly in $t \in [0, L]$, $x \in \mathbf{R}^n$:

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\alpha}} \int\limits_t^{t+\varepsilon^{\alpha}} A_{ij}\left(\frac{s}{\varepsilon}, x\right) ds = \frac{\partial X_{0i}}{\partial x_j} + \frac{\partial I_{0i}}{\partial x_j},$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\alpha+1}} \int\limits_t^{t+\varepsilon^{\alpha}} \int\limits_t^{t+\varepsilon^{\alpha}} B_{ij}\left(\frac{s}{\varepsilon}, \frac{\tau}{\varepsilon}, x\right) ds d\tau = g_{ij}(x),$$

where $\frac{1}{3} < \alpha < \frac{1}{2}$ and the matrix $\{B_{ij}\}$ is the nonnegative square root of the matrix $\{C_{ij}\}$.

8) There exists a nonnegative, monotone nondecreasing function $\rho(t)$, $t \geq 0$, $\rho(+0) = 0$, such that the matrices $g(x) = \{g_{ij}(x)\}$ and $\{\frac{\partial(X_0+I_0)}{\partial x}\}$ satisfy the inequalities

$$|g(\bar{x}(s)) - g(\bar{x}(t))| \leq \rho(|t-s|)\,,$$

$$\left|\frac{\partial(X_0(\bar{x}(s)) + I_0(\bar{x}(s)))}{\partial x} - \frac{\partial(X_0(\bar{x}(t)) + I_0(\bar{x}(s)))}{\partial x}\right| \leq \rho(|t-s|)\,.$$

9) For arbitrary $\tau \in [0, L]$, $\Delta > 0$, $\tau + \Delta \leq L$, we have

$$\begin{gathered}\left|\int\limits_{\tau}^{\tau+\Delta}\left[X_0(\bar{x}(s)) - f\left(\frac{s}{\varepsilon}, \bar{x}(s)\right)\right]ds\right| \leq C\Delta\varepsilon, \\ \left|\int\limits_{\tau}^{\tau+\Delta}\left[I_0(\bar{x}(s))ds - \varepsilon\sum_{\tau<\tau_i<\tau+\Delta}\mathbf{E}I_i(\bar{x}(\tau_i), w)\right]\right| \leq C\Delta\varepsilon\,.\end{gathered} \tag{5.34}$$

Lemma 5.2. *Let all the above conditions hold and $\beta(\tau) = o(\tau^{-\lambda})$ with $\lambda > 16$ for $\tau \to \infty$.*

Then, as $\varepsilon \to 0$, the process $Y^{(\varepsilon)}(\frac{\tau}{\varepsilon})$ converges on $[0, L]$ in the sense of weak convergence of measures to a solution of the linear diffusion type stochastic equation

$$dX^{(0)}(\tau) = \frac{\partial(X_0(\bar{x}(\tau)) + I_0(\bar{x}(\tau))}{\partial x}\,d\tau + g(\bar{x}(\tau))dW(\tau), \qquad X^{(0)}_{(0)} = 0\,, \tag{5.35}$$

where $W(\tau)$ is an n-dimensional Wiener process.

A proof of this lemma is directly obtained from Theorem 2 in [26].

We use Lemmas 5.1 and 5.2 to prove the following result.

Theorem 5.2. *Let the above conditions* 1)–9) *and* a)–d) *hold. Then the process $X^{(\varepsilon)}(\frac{\tau}{\varepsilon})$ weakly converges on $[0, L]$ (in the sense of weak convergence of measures), as $\varepsilon \to 0$, to a solution of problem (5.35).*

Proof. To make the calculations simpler, we will carry them out for $\mathbf{R}$. The case $\mathbf{R}^n$ is treated similarly. First of all, note that the finite dimensional distributions of $X^{(\varepsilon)}(\frac{\tau}{\varepsilon})$ converge to the corresponding finite dimensional distributions of $X^{(0)}(\tau)$. This follows from Lemmas 5.1 and 5.2. Since the realizations of the process $X^{(\varepsilon)}(\frac{\tau}{\varepsilon})$ belongs to $D[0,L]$, which denotes A. V. Skorokhod's space that is the space of functions having no discontinuities of the second kind, to show the weak convergence it is sufficient to prove that the family of measures generated by the processes $X^{(\varepsilon)}(\frac{\tau}{\varepsilon})$ in the space $D[0,L]$ is weakly compact. To this end, see [52, p. 508], it is sufficient to prove existence of $\alpha > 0$, $\beta > 0$, $H > 0$ such that the following inequality holds for all τ, c_1, c_2 ($\tau + c_1 < \tau + c_2 \in [0,L]$) and $\varepsilon > 0$:

$$\mathbf{E}[|X^{(\varepsilon)}(\tau+c_1) - X^{(\varepsilon)}(\tau)||X^{(\varepsilon)}(\tau+c_2) - X^{(\varepsilon)}(\tau+c_1)|]^{\alpha} \le Hc_2^{1+\beta}. \quad (5.36)$$

Let $k \in N$. By the Hölder inequality, we have

$$\begin{aligned}&\mathbf{E}[|X^{(\varepsilon)}(\tau+c_1) - X^{(\varepsilon)}(\tau)|^k|X^{(\varepsilon)}(\tau+c_2) - X^{(\varepsilon)}(\tau+c_1)|^k]\\ &\quad \le \mathbf{E}^{\frac{1}{2}}|X^{(\varepsilon)}(\tau+c_1) - X^{(\varepsilon)}(\tau)|^{2k}\mathbf{E}^{\frac{1}{2}}X^{(\varepsilon)}(\tau+c_2) - X^{(\varepsilon)}(\tau+c_1)|^{2k}. \quad (5.37)\end{aligned}$$

Let us estimate each factor in (5.37). Since

$$x(t,x_0) = x_0 + \varepsilon\int_0^t X(s,x(s,x_0),w)ds + \varepsilon\sum_{0<t_i<t} I_i(x(t_i,x_0),w),$$

we have

$$\begin{aligned}&\mathbf{E}|X^{(\varepsilon)}(\tau+c_1) - X^{(\varepsilon)}(\tau)|^{2k}\\ &= \frac{1}{\varepsilon^k}\mathbf{E}\left|\int_\tau^{\tau+c_1} X\left(\frac{s}{\varepsilon}, x\left(\frac{s}{\varepsilon}\right), w\right)ds + \varepsilon\sum_{\tau<\tau_i<\tau+c_1} I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right)\right.\\ &\quad \left. - \int_\tau^{\tau+c_1}[X_0(\bar{x}(s) + I_0(\bar{x}(s))]ds\right|^{2k}\\ &= \frac{1}{\varepsilon^k}\mathbf{E}\left|\int_\tau^{\tau+c_1}\left[X\left(\frac{s}{\varepsilon}, x\left(\frac{s}{\varepsilon}\right), w\right) - f\left(\frac{s}{\varepsilon}, x\left(\frac{s}{\varepsilon}\right)\right)\right]ds\right.\\ &\quad + \varepsilon\sum_{\tau<\tau_i<\tau+c_1}\left[I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right) - \mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right)\right]\\ &\quad + \int_\tau^{\tau+c_1}\left[f\left(\frac{s}{\varepsilon}, x\left(\frac{s}{\varepsilon}\right)\right) - f\left(\frac{s}{\varepsilon}, \bar{x}(s)\right)\right]ds - \int_\tau^{\tau+c_1}\left[X_0(\bar{x}(s)) - f\left(\frac{s}{\varepsilon}, x(s)\right)\right]ds\end{aligned}$$

$$+ \varepsilon \sum_{\tau<\tau_i<\tau+c_1} \left[\mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right) - \mathbf{E}I_i(\bar{x}(\tau_i), w)\right]$$

$$- \left[\int\limits_{\tau}^{\tau+c_1} I_0(\bar{x}(s))ds - \varepsilon \sum_{\tau<\tau_i<\tau+c_1} \mathbf{E}I_i(\bar{x}(\tau_i), w)\right]\Bigg|^{2k}$$

$$\leq \frac{6^{2k-1}}{\varepsilon^k}\left[\mathbf{E}\left|\int\limits_{\tau}^{\tau+c_1}\left(X\left(\frac{s}{\varepsilon}, x\left(\frac{s}{\varepsilon}\right), w\right) - f\left(\frac{s}{\varepsilon}, x\left(\frac{s}{\varepsilon}\right)\right)\right)ds\right|^{2k}\right.$$

$$+ \varepsilon^{2k}\mathbf{E}\left|\sum_{\tau<\tau_i<\tau+c_1}\left[I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right) - \mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right)\right]\right|^{2k}$$

$$+ \mathbf{E}\left|\int\limits_{\tau}^{\tau+c_1}\left[f\left(\frac{s}{\varepsilon}, x\left(\frac{s}{\varepsilon}\right)\right) - f\left(\frac{s}{\varepsilon}, \bar{x}(s)\right)\right]ds\right|^{2k}$$

$$+ \left|\int\limits_{\tau}^{\tau+c_1}\left[X_0(\bar{x}(s)) - f\left(\frac{s}{\varepsilon}, \bar{x}(s)\right)\right]ds\right|^{2k}$$

$$+ \varepsilon^{2k}\mathbf{E}\left|\sum_{\tau<\tau_i<\tau+c_1}\left|\mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right) - \mathbf{E}I_i(\bar{x}(\tau_i), w)\right]\right|^{2k}$$

$$\left. + \left[\int\limits_{\tau}^{\tau+c_1} I_0(\bar{x}(s))ds - \varepsilon \sum_{\tau<\tau_i<\tau+c_1} \mathbf{E}I_i(\bar{x}(\tau_i), w)|^{2k}\right]\right. .$$

Let us now estimate each term in the latter inequality. By Lemma 2 in [26], we have

$$\mathbf{E}\left|\int\limits_{\tau}^{\tau+c_1}\left[X\left(\frac{s}{\varepsilon}, x\left(\frac{s}{\varepsilon}\right), w\right) - f\left(\frac{s}{\varepsilon}, x\left(\frac{s}{\varepsilon}\right)\right)\right]ds\right|^{2k} \leq c_2(k)\varepsilon^k c_1^k\,. \tag{5.38}$$

The expressions

$$\left|\int\limits_{\tau}^{\tau+c_1}\left[X_0(\bar{x}(s)) - f\left(\frac{s}{\varepsilon}, \bar{x}(s)\right)\right]ds\right|^{2k}$$

and

$$\left|\int\limits_{\tau}^{\tau+c_1} I_0(\bar{x}(s))ds - \varepsilon \sum_{\tau<\tau_i<\tau+c_1} \mathbf{E}I_i(\bar{x}(\tau_i), w)\right|^{2k}$$

by condition 9) admit the estimate with

$$C(k)c_1^k\varepsilon^{2k}\,. \tag{5.39}$$

In order to estimate the expression

$$\varepsilon^{2k}\mathbf{E}\left|\sum_{\tau<\tau_i<\tau+c_1}\left[I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right),w\right)-\mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right),w\right)\right]\right|^{2k},$$

introduce the function

$$\Phi\left(\frac{\tau}{\varepsilon},x\left(\frac{\tau}{\varepsilon}\right),w\right)=\begin{cases}\varepsilon\left[I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right),w\right)-\mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right),w\right)\right]_n, & \tau\in\left[\tau_i,\tau_i+\dfrac{1}{n}\right],\\ 0, & \text{in other points.}\end{cases} \tag{5.40}$$

Take n so large that the segments $[\tau_i,\tau_i+\frac{1}{n}]$ and $[\tau_{i+1},\tau_{i+1}+\frac{1}{n}]$ would not intersect. Then it is clear that

$$\int_{\tau}^{\tau+c_1}\Phi\left(\frac{s}{\varepsilon},x\left(\frac{s}{\varepsilon}\right),w\right)ds=\varepsilon\sum_{\tau<\tau_i<\tau+c_1}\left[I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right),w\right)-\mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right),w\right)\right].$$

Let us check that the function $\Phi(\frac{s}{\varepsilon},x(\frac{s}{\varepsilon}),w)$ satisfies the conditions of Lemma 2 in [26]. Condition 1) of this lemma evidently follows from the definition of the function Φ. Now, we have

$$\begin{aligned}\mathbf{E}|x(t)-x(s)|^k&=\mathbf{E}\left|\varepsilon\int_s^t X(u,x(u),w)du+\varepsilon\sum_{s<t_i<t}I_i(x(t_i),w)\right|^k\\&\leq 2^{k-1}\varepsilon^k\left[\mathbf{E}\left(\int_s^t|X(u,x(u),w)|du\right)^k+\mathbf{E}\left(\sum_{s<t_i<t}|I_i(x(t_i),w)|\right)^k\right]\\&\leq 2^{k-1}\varepsilon^k\left(\int_s^t\mathbf{E}|X(u,x(u),w)|^k du|t-s|^{k-1}\right.\\&\qquad\left.+\sum_{s<t_i<t}\mathbf{E}|I_i(x(t_i),w)|^kC^{k-1}|t-s|^{k-1}\right)\\&\leq 2^{k-1}\varepsilon^k(C|t-s|^k+C^{k+1}|t-s|^k),\end{aligned}$$

which shows that condition 2) of the lemma is satisfied. The third condition is implied by condition a)–d).

It is clear that the function $\Phi(\frac{s}{\varepsilon}, x(\frac{\tau}{\varepsilon}), w)$ is $F^{\frac{\tau_i}{\varepsilon}}_{\frac{\tau_i}{\varepsilon}}$-measurable for $\tau \in [\tau_i, \tau_i + \frac{1}{n}]$, which is sufficient for applying Lemma 2 in [26]. This gives the estimate

$$\varepsilon^{2k}\mathbf{E}\left|\sum_{\tau<\tau_i<\tau+c_1}\left[I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right) - \mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right)\right]\right|^{2k} \leq C(k)\varepsilon^k c_1^k . \quad (5.41)$$

Now we have

$$\mathbf{E}\left|\int_{\tau}^{\tau+c_1}\left[f\left(\frac{s}{\varepsilon}, x\left(\frac{s}{\varepsilon}\right)\right) - f\left(\frac{s}{\varepsilon}, \bar{x}(s)\right)\right]ds\right|^{2k} \leq \mathbf{E}\left(\int_{\tau}^{\tau+c_1} K\left|x\left(\frac{s}{\varepsilon}\right) - \bar{x}(s)\right|ds\right)^{2k}$$

$$\leq K^{2k}C_1^{2k-1}\int_{\tau}^{\tau+c_1}\mathbf{E}\left|x\left(\frac{s}{\varepsilon}\right) - \bar{x}(s)\right|^{2k} ds\,. \quad (5.42)$$

Since

$$\mathbf{E}\left|x\left(\frac{s}{\varepsilon}\right) - \bar{x}(s)\right|^{2k} = \mathbf{E}\left|\int_0^s X\left(\frac{u}{\varepsilon}, x(\frac{u}{\varepsilon}), w\right)du\right.$$

$$\left. + \varepsilon\sum_{0<\tau_i<s} I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right) - \int_0^s[X_0(\bar{x}(u)) + I_0(\bar{x}(u))]du\right|^{2k}$$

$$= \mathbf{E}\left|\int_0^s\left[X\left(\frac{u}{\varepsilon}, x\left(\frac{u}{\varepsilon}\right), w\right) - f\left(\frac{u}{\varepsilon}, x\left(\frac{u}{\varepsilon}\right)\right)\right]du\right.$$

$$+ \int_0^s\left[f\left(\frac{u}{\varepsilon}, x\left(\frac{u}{\varepsilon}\right)\right) - f\left(\frac{u}{\varepsilon}, \bar{x}(u)\right)\right]du - \int_0^s\left[X_0(\bar{x}(u)) - f\left(\frac{u}{\varepsilon}, \bar{x}(u)\right)\right]du$$

$$+ \varepsilon\sum_{0<\tau_i<s}\left[I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right) - \mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right)\right]$$

$$+ \varepsilon\sum_{0<\tau_i<s}\left[\mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right) - \mathbf{E}I_i(\bar{x}(\tau_i), w)\right]$$

$$\left. - \int_0^s\left[I_0(\bar{x}(u))du - \varepsilon\sum_{0<\tau_i<s}\mathbf{E}I_i(\bar{x}(\tau_i), w)\right]\right|^{2k}$$

$$\leq 6^{2k-1}\left[\mathbf{E}\left|\int_0^s\left[X\left(\frac{u}{\varepsilon}, x\left(\frac{u}{\varepsilon}\right), w\right) - f\left(\frac{u}{\varepsilon}, x\left(\frac{u}{\varepsilon}\right)\right)\right]du\right|^{2k}\right.$$

$$+\mathbf{E}\left|\int_0^s\left[f\left(\frac{u}{\varepsilon},x\left(\frac{u}{\varepsilon}\right)\right)-f\left(\frac{u}{\varepsilon},\bar{x}(u)\right)\right]du\right|^{2k}$$

$$+\left|\int_0^s\left[X_0(\bar{x}(u))-f\left(\frac{u}{\varepsilon},\bar{x}(u)\right)\right]du\right|^{2k}$$

$$+\varepsilon^{2k}\mathbf{E}\left|\sum_{0<\tau_i<s}\left[I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right),w\right)-\mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right),w\right)\right]\right|^{2k}$$

$$+\varepsilon^{2k}\mathbf{E}\left|\sum_{0<\tau_i<s}\left[\mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right),w\right)-\mathbf{E}I_i(\bar{x}(\tau_i),w)\right]\right|^{2k}$$

$$+\left.\left|\int_0^s\left[I_0(\bar{x}(u))du-\varepsilon\sum_{0<\tau_i<s}\mathbf{E}I_i(\bar{x}(\tau_i),w)\right]\right|^{2k}\right],$$

estimates (5.38), (5.39), and (5.41) show that the expression

$$\mathbf{E}\left|x\left(\frac{s}{\varepsilon}\right)-\bar{x}(s)\right|^{2k}$$

does not exceed

$$6^{2k-1}\left[C(k)\varepsilon^k s^k+2C(k)s^{2k}\varepsilon^{2k}+C(k)\varepsilon^k s^k\right.$$

$$+\mathbf{E}\left|\int_0^s\left[f\left(\frac{u}{\varepsilon},x\left(\frac{u}{\varepsilon}\right)\right)-f\left(\frac{u}{\varepsilon},\bar{x}(u)\right)\right]du\right|^{2k}$$

$$\left.+\varepsilon^{2k}\mathbf{E}\left|\sum_{0<\tau_i<s}\left[\mathbf{E}I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right),w\right)-\mathbf{E}I_i(\bar{x}(\tau_i),w)\right]\right|^{2k}\right]$$

$$\le 6^{2k-1}\left[\bar{C}\varepsilon^k s^k+K^{2k}s^{2k-1}\int_0^s\mathbf{E}\left|x\left(\frac{u}{\varepsilon}\right)-\bar{x}(u)\right|^{2k}du\right.$$

$$\left.+K^{2k}\varepsilon^{2k}i^{2k-1}(s)\sum_{0<\tau_i<s}\mathbf{E}\left|x\left(\frac{\tau_i}{\varepsilon}\right)-\bar{x}(\tau_i)\right|^{2k}\right],$$

where $i(s)$ is the number of points of the impulsive effects on the segment $[0,s]$, which admits an estimate with $C\frac{L}{\varepsilon}$.

Then we have

$$\varepsilon^{2k} K^{2k} i^{2k-1}(s) \le q\varepsilon\,,$$

where q is a constant that does not depend on s and ε.

Thus,

$$\mathbf{E}\left|x\left(\frac{s}{\varepsilon}\right) - \bar{x}(s)\right|^{2k} \le C_3 \varepsilon^k s^k + C_4 \int\limits_0^s \mathbf{E}\left|x\left(\frac{u}{\varepsilon}\right) - \bar{x}(u)\right|^{2k} du$$
$$+ q\varepsilon \sum_{0<\tau_i<s} \mathbf{E}\left|x\left(\frac{\tau_i}{\varepsilon}\right) - \bar{x}(\tau_i)\right|^{2k},$$

which, with a use of the generalized Gronwall-Bellman inequality, gives

$$\mathbf{E}\left|x\left(\frac{s}{\varepsilon}\right) - \bar{x}(s)\right|^{2k} \le C_5 \varepsilon^k \prod_{0<\tau_i<s} (1+q\varepsilon) = \varepsilon^k C_5 (1+q\varepsilon)^{i(s)} = \varepsilon^k C_5 (1+q\varepsilon)^{\frac{q_1}{\varepsilon}}\,,$$

where q_1 is a constant independent of ε. Thus

$$\varepsilon^k C_6 (e^{qq_1} + O(\varepsilon)) \le \varepsilon^k C_7, \tag{5.43}$$

for sufficiently small ε. By substituting (5.43) into (5.42) we obtain

$$K^{2k} C_1^{2k-1} \int\limits_{\tau}^{\tau+c_1} \mathbf{E}\left|x\left(\frac{s}{\varepsilon}\right) - \bar{x}(s)\right|^{2k} \le \varepsilon^k C_8 C_1^{2k}\,. \tag{5.44}$$

The term

$$\varepsilon^{2k}\mathbf{E}\left|\sum_{\tau<\tau_i<\tau+c_1} \left[\mathbf{E} I_i\left(x\left(\frac{\tau_i}{\varepsilon}\right), w\right) - \mathbf{E} I_i(\bar{x}(\tau_i), w)\right]\right|^{2k} \le \varepsilon^k C_9 C_1^{2k}\,, \tag{5.45}$$

can be estimated in the same way replacing s with τ_i. It follows from (5.38), (5.39), (5.41), (5.44), and (5.45) that

$$\mathbf{E}|X^{(\varepsilon)}(\tau + c_1) - X^{(\varepsilon)}(\tau)|^{2k} \le \frac{1}{\varepsilon^k}(R\varepsilon^k C_1^k + 2\varepsilon^{2k} C^{2k} C_1^{2k}$$
$$+ C(k)\varepsilon^k C_1^k + \varepsilon^k C_8 C_1^{2k}) \le C_9(k) C_1^k\,. \tag{5.46}$$

Estimating the second factor in formula (5.37) we similarly get

$$\mathbf{E}|X^{(\varepsilon)}(\tau + c_2) - X^{(\varepsilon)}(\tau + C_1)|^{2k} \le C_{10}(C_2 - C_1)^k\,. \tag{5.47}$$

The above gives an estimate of the left-hand side of (5.37) with the quantity

$$C_{11}C_1^{\frac{k}{2}}(C_2-C_1)^{\frac{k}{2}}.$$

Set $k=2$. The latter relation is estimated with $C_{11}C_2^2$, which is sufficient for inequality (5.36) to hold for $\alpha=2$ and $\beta=1$. This proves the theorem. $\square$

Let us make a few remarks that would facilitate a verification of the conditions of the theorem.

First, if for fixed x, $X(t,x,w)$ and $I_i(x,w)$ are periodic in the generalized sense, that is, there exist $\theta>0$ and $p\in N$ such that

$$\begin{gathered} f(t,x)=f(t+\theta,x), C_{ij}(t,s,x)=C_{ij}(t+\theta,s+\theta,x)\,, \\ t_{i+p}-t_i=\theta, \quad \mathbf{E}I_{i+p}(x,w)=\mathbf{E}I_i(x,w) \quad \forall i\in N\,, \end{gathered} \tag{5.48}$$

then conditions (5.3) of Theorem 5.1 are evidently satisfied, and we have

$$\frac{\partial X_{0i}(x)}{\partial x_j}+\frac{\partial I_{0i}(x)}{\partial x_j}=\frac{1}{\theta}\int\limits_0^\theta \mathbf{E}\frac{\partial X_i(t,x,w)}{\partial x_j}dt+\frac{\partial I_0(x)}{\partial x_j}\,, \tag{5.49}$$

$$I_0(x)=\frac{1}{\theta}\sum_{i=1}^p \mathbf{E}I_i(x,w)\,. \tag{5.50}$$

If the periodicity condition is satisfied, a verification of conditions 7)–9) becomes easier.

Indeed, since, by Lemma 2.1 in [131],

$$|\mathbf{E}\xi\eta-\mathbf{E}\xi\mathbf{E}\eta|<C[\beta(\tau)]^{\frac{\delta}{\delta+2}} \tag{5.51}$$

for arbitrary random variables ξ, which is F_0^t-measurable, and η, which is $F_{t+\tau}^\infty$-measurable, satisfying $\mathbf{E}|\xi|^{2+\delta}<C$ and $\mathbf{E}|\eta|^{2+\delta}<C$, it follows from condition 5) and (5.51) for all t and s that

$$|C_{ij}(t,s,x)|\le C[\beta(|t-s|]^{\frac{\delta}{\delta+2}}.$$

However, $\int\limits_t^{t+T} ds \int\limits_{-\infty}^{\infty} [\beta(|u-s|]^{\frac{\delta}{\delta+2}}du$ is convergent, since $\beta(\tau)=O(\tau^{-\lambda})$, $\lambda>16$. This proves the relation

$$\left|\int\limits_t^{t+T} ds\int\limits_t^{t+T} duC_{kj}(u,s,x)-\int\limits_t^{t+T} ds\int\limits_{-\infty}^{\infty} duC_{kj}(u,s,x)\right|<C\,. \tag{5.52}$$

Hence,

$$\frac{1}{T}\int\limits_{t}^{t+T}\int\limits_{t}^{t+T} C_{kj}(u,s,x)duds$$

$$= \frac{1}{T}\Bigg(\int\limits_{t}^{t+T} ds \int\limits_{t}^{t+T} du C_{kj}(u,s,x) - \int\limits_{t}^{t+T}\int\limits_{-\infty}^{\infty} C_{kj}(u,s,x)du\Bigg)$$

$$+ \frac{1}{T}\int\limits_{t}^{t+T} ds \int\limits_{-\infty}^{\infty} C_{kj}(u,s,x)du\,. \tag{5.53}$$

It follows from (5.52) and (5.53) that

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_{t}^{t+T}\int\limits_{t}^{t+T} C_{kj}(u,s,x)duds = \frac{1}{\theta}\int\limits_{0}^{\theta} ds \int\limits_{-\infty}^{\infty} C_{kj}(u,s,x)du\,, \tag{5.54}$$

and, hence, there exists a matrix $\{g_{ij}(x)\}$ that is the nonnegative square root of the matrix (5.54).

Condition 9) is important in Theorem 5.2. Let us give sufficient conditions that are simpler to verify.

Take arbitrary $\tau \in [0, L]$, $\Delta > 0$, $\tau + \Delta \leq L$, and $\varepsilon > 0$. Consider the function

$$\Psi(t, x(\frac{t}{\varepsilon})) = \begin{cases} MI_i\Big(x\Big(\dfrac{t_i}{\varepsilon}\Big), w\Big)n, & \text{for } t \in \Big[t_i, t_i + \dfrac{1}{n}\Big], \\ 0, & \text{in other points} \end{cases} \tag{5.55}$$

on the interval $[\frac{\tau}{\varepsilon}, \frac{\tau+\Delta}{\varepsilon}]$ for some i. Here t_i are points of the impulsive effects in system (5.1) on the interval $[\frac{\tau}{\varepsilon}, \frac{\tau+\Delta}{\varepsilon}]$. Choose $n \in N$ such that the segments $[t_i, t_i + \frac{1}{n}]$ and $[t_{i+1}, t_{i+1} + \frac{1}{n}]$ would not intersect for each n, and $(t_{i+1} - t_i)/2 \geq \frac{1}{n}$. This can be done because of the periodicity conditions (5.48). We will also assume that the difference between the point $\frac{\tau+\Delta}{\varepsilon}$ and the closest left point of the impulsive effect is not less than $\frac{1}{n}$.

Then it is clear that

$$\int\limits_{\tau}^{\tau+\Delta} \psi\Big(\frac{s}{\varepsilon}, x\Big(\frac{s}{\varepsilon}\Big)\Big)ds = \varepsilon \sum_{\tau<\tau_i<\tau+\Delta} \mathbf{E}I_i\Big(x\Big(\frac{\tau_i}{\varepsilon}\Big), w\Big) \tag{5.56}$$

and, hence,

$$\int_{\tau}^{\tau+\Delta} I_0(\bar{x}(s))ds - \varepsilon \sum_{\tau<\tau_i<\tau+\Delta} \mathbf{E} I_i(\bar{x}(\tau_i), w) = \int_{\tau}^{\tau+\Delta} \left[I_0(\bar{x}(s)) - \psi\left(\frac{s}{\varepsilon}, \bar{x}(s)\right)\right] ds\,.$$

Note now that periodicity conditions (5.48) also imply that the function $\psi(t,x)$ is periodic in t and that

$$I_0(x) = \frac{1}{\theta}\int_0^{\theta} \psi(t,x)dt. \tag{5.57}$$

Let the functions$f(t,x)$ and $\frac{\partial}{\partial x_k} f(t,x)$ satisfy the Hölder conditions in t uniformly with respect to x that belongs to the curve $x = \bar{x}(\tau)$, $0 \le \tau \le L$, where, as before, $\bar{x}(\tau)$ is a solution of the averaged equation (5.33).

Let, finally, the following conditions hold: there exist $Q > 0$, $0 < \alpha < 1$ such that

$$\begin{gathered} |\mathbf{E} I_i(\bar{x}(\tau), w) - \mathbf{E} I_j(\bar{x}(\tau), w)| \le Q|t_i - t_j|^{\alpha}, \\ \left|\frac{\partial}{\partial x_k}(\mathbf{E} I_i(\bar{x}(\tau), w) - \mathbf{E} I_j(\bar{x}(\tau), w))\right| \le Q|t_i - t_j|^{\alpha} \end{gathered} \tag{5.58}$$

uniformly in $0 \le \tau \le L$ for arbitrary moments of the impulsive effects, t_i, t_j. Inequalities (5.58) imply that the supports of $\psi(t,x)$ and $\frac{\partial \psi(t,x)}{\partial x_k}$ satisfy Hölder conditions uniformly with respect to $0 \le \tau \le L$.

Indeed, take arbitrary $s, t \in \operatorname{supp}\psi$. Let $s \in [t_i, t_i + \frac{1}{n}]$ and $t \in [t_k, t_k + \frac{1}{n}]$, and let there be l points $t_{i+1} \dots t_{i+l}$ of the impulsive effects between the points t_i and t_k with $t_{i+l+1} = t_k$. Then, by (5.55), we have

$$|\psi(t, \bar{x}(u)) - \psi(s, \bar{x}(u))| = n|\mathbf{E} I_k(\bar{x}(u), w) - \mathbf{E} I_i(\bar{x}(u), w)| \le Q|t_k - t_i|^{\alpha}\,.$$

Since

$$t - s \ge t_{i+1} - t_i - \frac{1}{n} + t_k - t_{i+1} = t_k - t_i - \frac{1}{n} \ge \frac{t_k - t_i}{2}\,,$$

we have

$$Qn|t_k - t_i|^{\alpha} = \eta\left|\frac{t_k - t_i}{2}\right| \le \eta|t - s|^{\alpha}\,,$$

where $\eta = 2^{\alpha} Qn$.

However, the Fourier coefficients of the functions $f(t,x) - X_0(x)$ and $\psi(t,x) - I_0(x)$, and their derivatives satisfy the estimate

$$\max\left\{|F_n(x)|, \left|\frac{\partial F_n(x)}{\partial x_k}\right|, |\psi_n(x)|, \left|\frac{\partial \psi_n(x)}{\partial x_k}\right|\right\} < \frac{c}{n^{\alpha}}\,. \tag{5.59}$$

It also follows from (5.49) and (5.50) that $F_0(x) = \psi_0(x) = 0$. Represent $f(t,x) - X_0(x)$ with its Fourier series,

$$f(t,x) - X_0(x) = \sum_{n \neq 0} F_n(x) e^{i\frac{2\pi t n}{\theta}},$$

and integrate it to obtain

$$\left| \int_{\tau}^{\tau+\Delta} \left[f\left(\frac{s}{\varepsilon}, \bar{x}(s)\right) - X_0(\bar{x}(s)) \right] ds \right| = \left| \int_{\tau}^{\tau+\Delta} \sum_{n \neq 0} F_n(\bar{x}(s)) e^{\frac{i2\pi n}{\theta}\frac{s}{\varepsilon}} ds \right|$$

$$= \left| \sum_{n \neq 0} \frac{\theta\varepsilon}{i2\pi n} e^{\frac{i2\pi n}{\theta}\frac{s}{\varepsilon}} F_n(\bar{x}(s))\Big|_{\tau}^{\tau+\Delta} - \frac{\theta\varepsilon}{i2\pi n} \int_{\tau}^{\tau+\Delta} \frac{\partial F_n(\bar{x}(s))}{\partial x} \big[X_0(\bar{x}(s)) \right.$$

$$\left. + I_0(\bar{x}(s)) e^{\frac{i2\pi n}{\theta}\frac{s}{\varepsilon}} ds \big] \right|.$$

However, by (5.59), the latter expression does not exceed

$$\varepsilon \sum_{n \neq 0} \left[\left| \frac{c\theta}{i2\pi n^{\alpha+1}} \right| + \left| \frac{\rho}{i2\pi n^{\alpha+1}} \right| \right], \tag{5.60}$$

where ρ is some constant independent of n and ε. Since series (5.60) converges, the first inequality in condition 9) is satisfied. In the same way, one can check that the second condition is also satisfied.

By summarizing the above results, we get the following.

Theorem 5.3. *Let conditions 1), 2), 3) of the preceding section be satisfied, together with conditions 4)–6), 8), a)–d) and* (5.22), (5.58) *where* $\beta(\tau) = O(\tau^{-\lambda})$, $\lambda \geq 16$.

If the processes $X(t,x,w)$ *and* $I_i(x,w)$ *are periodic in the wide sense, that is, they satisfy conditions* (5.48), *and* $f(t, \overline{x(\tau)})$, *together with its partial derivatives with respect to* x, *satisfy the Hölder condition in* t *uniformly with respect to* $0 \leq \tau \leq L$, *then Theorem 5.2 and formulas* (5.49), (5.50), (5.54) *hold true.*

5.3 Applications to the theory of nonlinear oscillations

In the preceding section, we have shown that, under the mentioned conditions, the normed difference between an exact solution and the corresponding solution of the averaged system weakly converge, as $\varepsilon \to 0$, to a solution of a diffusion type linear stochastic equation. It turns out that the behavior of the asymptotics of this difference can be described with somewhat different stochastic differential equation. To this end, let us again consider systems (5.32) and (5.33).

Theorem 5.4. *Let conditions 1), 2),* (5.21), *the fist condition in* (5.3), *and condition* (5.22) *be satisfied. Let us also assume the following.*

1) *There exist $c > 0$, $K > 0$ such that $\mathbf{E}|X(t,x,w)|^{6+3\delta} + \mathbf{E}|I_i(x,w)| \le c$ for all $t \ge 0$, $x \in \mathbf{R}^n$, $i \in \mathbf{N}$ and some $\delta > 0$, and let the following relations hold with probability* 1*:*

$$\left|\frac{\partial}{\partial x}[X(t,x,w) + I_0(x)]\right| < c\,,$$

$$\left|\frac{\partial^2}{\partial x_j \partial x_k}[X_\nu(t,x,w) + I_{o\nu}(x)]\right| < c, \qquad \nu, j, k = \overline{1,n}\,,$$

$$|X(t,x',w) - X(t,x'',w)| + |I_i(x',w) - I_i(x'',w)| \le K|x' - x''|$$

$\forall t \ge 0,\ x, x', x'' \in \mathbf{R}^n,\ i = 1, 2, ...$

2) *The limit*

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_t^{t+T} \int\limits_t^{t+T} ds d\tau \mathbf{E}[X_k(\tau,x,w) - \mathbf{E}X_k(\tau,x,w)] \times [X_i(s,x,w) - \mathbf{E}X_i(s,x,w)] = A_{ki}(x)\,. \tag{5.61}$$

exists uniformly in $x \in \mathbf{R}^n$, $t \ge 0$.

3) *For all $\tau \in [0, L]$,*

$$\left|\int\limits_0^\tau \left[\mathbf{E}X\left(\frac{s}{\varepsilon}, \bar{x}(s), w\right) - X_0(\bar{x}(s))\right] ds\right| \le c\varepsilon, \tag{5.62}$$

$$\left|\int\limits_0^\tau \left[\mathbf{E}\frac{\partial X_k}{\partial x_j}\left(\frac{s}{\varepsilon}, \bar{x}(s), w\right) - \frac{\partial X_{0k}}{\partial x_j}(\bar{x}(s))\right] ds\right| \le c\varepsilon\,, \tag{5.63}$$

$$\left|\int\limits_0^\tau I_0(\bar{x}(s)) ds - \varepsilon \sum_{0<\tau_i<\tau} \mathbf{E}I_i(\bar{x}(\tau_i), w)\right| \le c\varepsilon\,, \tag{5.64}$$

$$\left| \int_0^\tau \frac{\partial I_{0k}(\bar{x}(s))}{\partial x_j} ds - \varepsilon \sum_{0<\tau_i<\tau} \mathbf{E} \frac{\partial I_{ik}}{\partial x_j}(\bar{x}(\tau_i), w) \right| \le c\varepsilon\,, \tag{5.65}$$

where $x = \bar{x}(s)$ is a solution of the averaged equation (5.33), $k, i, j = \overline{1, n}$.

If conditions a)–d) in the preceding section are satisfied with $\beta(\tau) = O(\tau^{-\lambda})$, $\lambda > 16$, then the process $X^{(\varepsilon)}(\frac{\tau}{\varepsilon})$ weakly converges as $\varepsilon \to 0$, on the segment $[0, L]$, to a Gaussian Markov process $X^{(0)}(\tau)$ that satisfies the linear stochastic system

$$X^{(0)}(\tau) = \int_0^\tau \frac{\partial (X_0(\bar{x}(s)) + I_0(\bar{x}(s)))}{\partial x} X^{(0)}(s) ds + W^{(0)}(\tau)\,, \tag{5.66}$$

where $W^{(0)}(\tau)$ is a Gaussian process with independent increments, zero expectation, and the correlation matrix

$$\int_0^\tau A_{ij}(\bar{x}(s)) ds\,. \tag{5.67}$$

Proof. First of all note that the conditions of the theorem imply that the conditions of Lemma 5.1 are verified. Now, it follows from [131] that condition d) in Section 2.2 implies that

$$\sup_{t\ge 0} \sup_{\xi,\eta} |\mathbf{E}\xi\eta - \mathbf{E}\xi\mathbf{E}\eta| = \alpha(\tau) \to 0, \qquad \tau \to \infty\,, \tag{5.68}$$

where $\sup\limits_{\xi,\eta}$ is taken over all ξ that are measurable with respect to F_0^t with $|\xi| < 1$, and all η, measurable with respect to $F_{t+\tau}^\infty$, $|\eta| < 1$, and we also have the estimate

$$\beta(\tau) \le \alpha(\tau) \le 16\beta(\tau)\,. \tag{5.69}$$

This is sufficient to apply Theorem 3.1 in [68] that implies that all conditions of Lemma 5.2 are satisfied. Hence, the finite dimensional distributions of the process $X^{(\varepsilon)}(\frac{\tau}{\varepsilon})$ converge to the corresponding finite dimensional distributions of the process $X^{(0)}(\tau)$. Weak compactness of the family of measures generated by the processes $X^{(\varepsilon)}(\frac{\tau}{\varepsilon})$ can be proved similarly to the corresponding fact in Theorem 5.2. □

Let now the right-hand sides of the systems under consideration be periodic random processes, i.e., the processes that satisfy condition (5.48). Let also the function $f(t, \bar{x(\tau)}) = \mathbf{E}X(t, \bar{x(\tau)}, w)$ and its derivatives with respect to x up to order two satisfy the Hölder condition in t uniformly with respect to $0 \le \tau \le L$. Let condition (5.58) be satisfied and the following inequality hold:

$$\left| \frac{\partial^2}{\partial x_\nu \partial x_k} (\mathbf{E} I_i(\bar{x}(\tau), w) - \mathbf{E} I_j(\bar{x}(\tau), w)) \right| \le Q|t_i - t_j|^\alpha, \tag{5.70}$$

where Q, α and t_i, t_j are the same as in condition (5.58), $\nu, k = \overline{1, n}$.

Then, as in Section 2.2, one can prove that the functions $f(x,t) - X_0(x)$, $\psi(t,x) - I_0(x)$, and their derivatives satisfy the estimates

$$\begin{aligned} &\max \left\{ |F_n(x)|, \left| \frac{\partial F_n(x)}{\partial x} \right|, \left| \frac{\partial^2 F_n(x)}{\partial x_k \partial x_i} \right| \right\} < \frac{c}{n^\alpha}, \\ &\max \left\{ |\psi_n(x)|, \left| \frac{\partial \psi_n(x)}{\partial x_k} \right|, \left| \frac{\partial^2 \psi_n(x)}{\partial x_k \partial x_i} \right| \right\} < \frac{c}{n^\alpha} \end{aligned} \tag{5.71}$$

that permit to obtain the following result.

Theorem 5.5. *Let conditions 1), 2) of Section 2.1 and conditions a)–d) of Theorem 5.4 hold. Let also $X(t,x,w)$, $I_i(x,w)$ be periodic in wide sense for each fixed x. Moreover, let the function $f(t, \bar{x}(\tau))$ and its derivatives up to order two with respect to x satisfy the Hölder condition in t uniformly with respect to* $0 \le \tau \le L$*, and let conditions* (5.58) *and* (5.70) *hold.*

Then Theorem 5.4 holds true with $X_0(x)$, $I_0(x)$, and $A_{kj}(x)$ expressed in terms of the respective formulas (5.49), (5.50), *and* (5.54)*.*

We now give some applications of the theorems obtained above.

1. Let $X(t,w)$ be a process stationary in the wide sense, with zero mean, and a correlation matrix $K_{kj}(\tau)$ satisfying condition a)–d) in Theorem 5.4. Let $\mathbf{E}|X(t,w)|^{6+3\delta} < \infty$ and $\mathbf{E} I_i(w) = 0$ for arbitrary $i \in \mathbf{N}$. Assume that the first conditions in (5.3) and (5.22) hold.

Then the limits

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} \int\limits_t^{t+T} \mathbf{E} X(s,x,w) ds = X_0(x), \\ &\lim_{T\to\infty} \frac{1}{T} \sum_{t<t_i<t+T} \mathbf{E} I_i(x,w) ds = I_0(x) \end{aligned} \tag{5.72}$$

exist uniformly in t and x.

This is sufficient to satisfy all conditions in Theorem 5.5 for the system

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon X(t,w), \qquad t \ne t_i, \\ \Delta x|_{t=t_i} &= \varepsilon I_i(\omega)\,. \end{aligned}$$

This theorem implies that finite dimensional distributions of the process

$$X^{(\varepsilon)}\left(\frac{\tau}{\varepsilon}\right) = \sqrt{\varepsilon}\left(\int\limits_0^{\frac{\tau}{\varepsilon}} X(t,w)dt + \sum_{t<t_i<\frac{\tau}{\varepsilon}} I_i(w)\right)$$

converge to finite dimensional distributions of the process $X^{(0)}(\tau)$ that is a Gaussian process with independent increments and the correlation matrix $A_{kj}(\tau) = \tau \int\limits_{-\infty}^{\infty} K_{kj}(s)ds$. A similar result for processes with continuous trajectories is obtained in [68].

2. Assume that the following conditions are satisfied: conditions 1), 2), the first condition in (5.3) in Section 2.1, conditions (5.22), a)–d) in Section 2.2, and condition 1) in Theorem 5.4. Moreover, assume that there exists a function $j(t)$ and a sequence $\{r_i\}$ such that

$$|\mathbf{E}X_k(t,x,w) - \beta_k(t,x)| < j(t)\,, \tag{5.73}$$

$$\left|\mathbf{E}\frac{\partial X_k(t,x,w)}{\partial x_j} - \frac{\partial \beta_k(t,x)}{\partial x_j}\right| < j(t)\,, \tag{5.74}$$

$$\begin{aligned} &|\mathbf{E}\{[X_k(t,x,w) - M\mathbf{E}_k(t,x,w)][X_j(s,x,w) - \mathbf{E}X_j(s,x,w)]\} \\ &\quad -\alpha_{kj}(t,s,x)| < j(t)j(s)\,, \end{aligned} \tag{5.75}$$

$$\begin{aligned} &\exists\theta > 0 : \beta_k(t+\theta,x) = \beta_k(t,s)\,, \\ &\alpha_{kj}(t+\theta,s+\theta,x) = \alpha_{kj}(t,s,x)\,, \end{aligned} \tag{5.76}$$

$$\int\limits_0^{\infty} j(t)dt < \infty\,. \tag{5.77}$$

Assume also that there is $p \in N$ such that the times of the impulsive effects in system (5.1) satisfy the conditions

$$t_{i+p} - t_i = \theta, \tag{5.78}$$

$$|\mathbf{E}I_i(x,w) - m_i(x)| < r_i\,, \tag{5.79}$$

$$\left|\mathbf{E}\frac{\partial I_{ik}(x,w)}{\partial x_j} - \frac{\partial m_{ik}(x)}{\partial x_j}\right| \le r_i\,, \tag{5.80}$$

where $\sum\limits_{i=0}^{\infty} r_i < \infty$ and $m_i(x) = m_{i+p}(x)$. Here $i \in N$, $k,j = \overline{1,n}$.

Theorem 5.6. *Let the conditions stated above be satisfied, and the function* $\beta_k(t, \bar{x}(\tau))$ *and its derivatives with respect to* x *up to order two satisfy the Hölder condition with respect to* t *uniformly in* $0 \le \tau \le L$. *Then the statement of Theorem 5.5 is valid for system* (5.32).

Proof. Indeed, the above conditions yield that

$$\frac{1}{T}\left|\int\limits_t^{t+T}[\mathbf{E}X_k(s,x,w) - \beta(s,x)]ds\right| \le \frac{1}{T}\int\limits_t^{t+T} j(s)ds \to 0\,,$$

for $T \to \infty$.

Moreover, we have

$$\frac{1}{T}\left|\int\limits_t^{t+T}\int\limits_t^{t+T}(\mathbf{E}\{[X_k(\tau,x,w) - \mathbf{E}X_k(\tau,x,w)][X_j(s,x,w) - \mathbf{E}X_j(j,x,w)]\}\right.$$

$$\left. - \alpha_{kj}(\tau,s,x))d\tau ds\right| \le \frac{1}{T}\int\limits_t^{t+T}\int\limits_t^{t+T} j(\tau)j(s)d\tau ds \to 0,\ T \to \infty\,,$$

and also

$$\frac{1}{T}\sum_{t<t_i<t+T}|\mathbf{E}I_i(x,w) - m_i(x)| \le \frac{1}{T}\sum_{t<t_i<t+T} r_i \to 0,\ T \to \infty\,,$$

which leads to

$$\begin{aligned} X_0(x) &= \frac{1}{\theta}\int\limits_0^\theta \beta(t,x)dt\,, \\ I_0(x) &= \frac{1}{\theta}\sum_{i=1}^p m_i(x)\,, \\ A_{kj}(x) &= \frac{1}{\theta}\int\limits_0^\theta\int\limits_{-\infty}^\infty \alpha_{ki}(t,s,x)dtds\,. \end{aligned} \tag{5.81}$$

It follows from conditions (5.73), (5.74), (5.79), (5.80) that for arbitrary $\tau \in [0, L]$,

$$\left|\int\limits_0^\tau[\mathbf{E}X\left(\frac{s}{\varepsilon},\bar{x}(s),w\right) - X_0(\bar{x}(s))]ds\right| \le \int\limits_0^\tau\left|\mathbf{E}X\left(\frac{s}{\varepsilon},\bar{x}(s),w\right) - \beta\left(\frac{s}{\varepsilon},\bar{x}(s)\right)\right|ds$$

$$+\int_0^{\tau}\left|\beta\left(\frac{s}{\varepsilon},\bar{x}(s)\right)-X_0(\bar{x}(s))\right|ds\le\varepsilon\int_0^{\frac{\tau}{\varepsilon}}|\mathbf{E}X(t,\bar{x}(\varepsilon t),w)-\beta(t,\bar{x}(\varepsilon t))|dt$$

$$+\int_0^{\tau}\left|\beta\left(\frac{s}{\varepsilon},\bar{x}(s)\right)-X_0(\bar{x}(s))ds\right|. \tag{5.82}$$

Since the function $\beta(t,x)$ satisfies the Hölder conditions, the second term in (5.82) does not exceed $c_1\varepsilon$ and, hence, we have the following estimate for (5.82):

$$\varepsilon\int_0^{\frac{\tau}{\varepsilon}}j(t)+c_1\varepsilon\le c\varepsilon\,.$$

Then (5.82) implies that condition (5.62) is satisfied. In the same way, we prove that conditions (5.63)–(5.65) are also satisfied.

Hence, all conditions of Theorem 5.4 are met, which finishes the proof of Theorem 5.5. □

As can be seen from the above, conditions of Theorem 5.4 are satisfied for a rather broad class of the processes $X(t,x,w)$ and $I_i(x,w)$ that converge in the sense of (5.73)–(5.80) to processes that are periodic in the wide sense and have the characteristics $\beta(t,x)$, $\alpha(t,s,x)$, and $m_i(x)$.

3. The results obtained above can be applied to study a random process in the output of a nonlinear device, which is close to a linear oscillator, with the input being a small periodic signal and a small random process and that is influenced by random impulsive effects. The case when the oscillator is described by an ordinary differential equation with random perturbations has been studied in detail in [68]. The case when the oscillator is described by a differential equation with impulsive effect is considered in [108].

Consider the equation

$$\begin{aligned}\ddot{x}+\mu^2x&=\varepsilon[f(\nu t,x,\dot{x})+\psi(x,\dot{x})\xi(t,w)],\qquad t\ne t_i,\\ \Delta x|_{t=t_i}&=\varepsilon[I_i(x,\dot{x})+J(x,\dot{x})\eta_i(w)],\end{aligned} \tag{5.83}$$

where $f(\nu t,x,\dot{x})$ is a sufficiently smooth function, periodic with period 2π in νt, $\xi(t)$ is a random process, stationary in the broad sense, having zero expectation and the correlation function $K(\tau)$, $\psi(x,\dot{x})$ is a sufficiently smooth function that defines the intensity of the "noise" at the point $(x,\dot{x})$. For the sake of simplicity, we assume that the period of the function $f(\nu t,x,\dot{x})$

coincides with the repetition period of impulsive effects, that is, the times of the impulsive effects, t_i and their values, $I_i(x, \dot{x})$, are such that $t_{i+p} - t_i = \frac{2\pi}{\nu}$, $I_{i+p}(x, \dot{x}) = I_i(x, \dot{x})$. Let the random variables η_i have expectation zero, $J(x, \dot{x})$ characterize the intensity of the random impulsive effect at the point $(x, \dot{x})$. With such a simplification, the frequency of both external effects, the impulsive and the continuous ones, is the same, ν.

Let μ and ν be incommensurable, and the points $\frac{2\pi k}{\mu}$, $k \in Z$, be not points of the discrete spectrum of the process $\xi(t)$. This case is considered as non-resonance.

It is known that, when studying oscillations in weakly nonlinear systems of type (5.83), it is convenient to pass from the Cartesian coordinates x, $\frac{dx}{dt}$ to the amplitude–phase coordinates a, φ given by

$$x = a \sin \varphi, \quad \frac{dx}{dt} = a\mu \cos \varphi, \quad \varphi = \mu t + \theta\,. \tag{5.84}$$

By first writing equation (5.83) with a use of Dirac's δ-function in the form

$$\begin{aligned} \ddot{x} + \mu^2 x = \varepsilon[f(\nu t, x, \dot{x}) + \psi(x, \dot{x})\xi(t, w)] \\ + \varepsilon \sum_{t<t_i<t} ([I_i(x, \dot{x}) + J(x, \dot{x})\eta_i]\delta(t - t_i))\,, \end{aligned} \tag{5.85}$$

and making the change of variables given in (5.84), we obtain the system

$$\begin{aligned} \frac{da}{dt} &= \frac{\varepsilon}{\mu}\bigg[f(\nu t, a \sin \varphi, a\mu \cos \varphi) + \psi(a \sin \varphi, a\mu \cos \varphi)\xi(t) \\ &\quad + \sum_{t<t_i<t} ((I_i(a \sin \varphi, a\mu \cos \varphi) + J(a \sin \varphi, a\mu \cos \varphi)\eta_i)\delta(t - t_i))\bigg] \cos \varphi\,, \\ \frac{d\varphi}{dt} &= \mu - \frac{\varepsilon}{a\mu}\bigg[f(\nu t, a \sin \varphi, a\mu \cos \varphi) + \psi(a \sin \varphi, a\mu \cos \varphi)\xi(t) \\ &\quad + \sum_{t<t_i<t} ((I_i(a \sin \varphi, a\mu \cos \varphi) + J(a \sin \varphi, a\mu \cos \varphi)\eta_i)\delta(t - t_i))\bigg] \sin \varphi\,, \end{aligned}$$

that can be rewritten as system (5.1) as follows:

$$\begin{aligned} \frac{da}{dt} &= \frac{\varepsilon}{\mu}[f(\nu t, a \sin \varphi, a\mu \cos \varphi) + \psi(a \sin \varphi, a\mu \cos \varphi)\xi(t)] \cos \varphi \\ \frac{d\varphi}{dt} &= \mu - \frac{\varepsilon}{a\mu}[f(\nu t, a \sin \varphi, a\mu \cos \varphi) + \psi(a \sin \varphi, a\mu \cos \varphi)\xi(t)] \sin \varphi, \quad t \neq t_i \end{aligned} \tag{5.86}$$

$$\Delta a|_{t=t_i} = \frac{\varepsilon}{\mu}[I_i(a\sin\varphi, a\mu\cos\varphi) + J(a\sin\varphi, a\mu\cos\varphi)\eta_i]\cos\varphi\,,$$

$$\Delta\varphi|_{t=t_i} = -\frac{\varepsilon}{a\mu}[I_i(a\sin\varphi, a\mu\cos\varphi) + J(a\sin\varphi, a\mu\cos\varphi)\eta_i]\sin\varphi\,.$$

Let us introduce the following notations:

$$f^{(1)}(\nu t, a, \varphi, w) = [f(\nu t, a\sin\varphi, a\mu\cos\varphi) + \psi(a\sin\varphi, a\mu\cos\varphi)\xi(t)]\cos\varphi\,,$$

$$f^{(2)}(\nu t, a, \varphi, w) = [f(\nu t, a\sin\varphi, a\mu\cos\varphi) + \psi(a\sin\varphi, a\mu\cos\varphi)\xi(t)]\sin\varphi\,,$$

$$I_i^{(1)}(a, \varphi, w) = [I_i(a\sin\varphi, a\mu\cos\varphi) + J(a\sin\varphi, a\mu\cos\varphi)\eta_i(w)]\cos\varphi\,,$$

$$I_i^{(2)}(a, \varphi, w) = [I_i(a\sin\varphi, a\mu\cos\varphi) + J(a\sin\varphi, a\mu\cos\varphi)\eta_i(w)]\sin\varphi.$$

Assume that the functions $I_i^{(1)}(a, \varphi, w)$, $I_i^{(2)}(a, \varphi, w)$ are finite trigonometric polynomials with respect to φ, and let

$$\frac{\partial I_i^{(j)}}{\partial\varphi} = \sum_{k=1}^{N}(A_{i_k}^{(j)}(a, w)\sin k\varphi + B_{i_k}^{(j)}(a, w)\cos k\varphi), \quad i = \overline{1, p}, \quad j = 1, 2\,.$$

Denote by $z_j(a, \varphi, t, w)$, $j = 1, 2$, the function

$$\begin{aligned} z_j(a, \varphi, t, w) = -\frac{1}{\pi}\sum_{k=1}^{N}\sum_{i=1}^{p}\Bigg\{&[A_{i_k}^{(j)}(a, w)\sin k\varphi \\ &+ B_{i_k}^{(j)}(a, w)\cos k\varphi]\sum_{k=1}^{\infty}\frac{\cos n\nu(t - t_i)}{(kw)^2 - n^2} + kw[B_{i_k}^{(j)}(a, w)\sin k\varphi \\ &- A_{i_k}^{(j)}(a, w)\cos k\varphi]\sum_{k=1}^{\infty}\frac{\sin n\nu(t - t_i)}{n[(kw)^2 - n^2]}\Bigg\}. \end{aligned} \tag{5.87}$$

A direct verification shows that these functions satisfy the relation

$$\frac{\partial z_j}{\partial\varphi}\mu + \frac{\partial z_j}{\partial t}\nu = -\sum_{i=1}^{p}\frac{\partial I_i^{(j)}(a, \varphi, w)}{\partial\varphi}\left\{\frac{1}{2} - \frac{\nu(t - t_i)}{2\pi}\right\}$$

with probability 1 for $t \neq t_i$, where $\{\frac{1}{2} - \frac{t}{2\pi}\}$ denotes a periodic function, with period 2π, that is defined over the period by $\frac{1}{\pi}\frac{\pi - t}{2}$, $0 < t < 2\pi$. In order to obtain an equation for first order approximations, let us make a change of variables in (5.86) by

$$\begin{aligned} a &= b + \frac{\varepsilon}{\mu}u^{(1)}(b, \theta_0, t, w), \\ \varphi &= \theta_0 - \frac{\varepsilon}{\mu b}u^{(2)}(b, \theta_0, t, w)\,, \end{aligned} \tag{5.88}$$

where

$$u^{(j)}(b,\theta_0,t,w) = \frac{1}{\mu}\int\int\left[f^{(j)}(\nu t,a,\varphi,w) + \frac{\nu}{2\pi}\sum_{i=1}^{p} I_i^{(j)}(\theta_0,w) - f_0^{(j)}(b)\right.$$

$$\left. - \frac{\nu}{2\pi}\sum_{i=1}^{p} I_i^{(j)}(b)\right] dt d\varphi + z_j(b,\theta_0,t,w)$$

$$+ \sum_{i=1}^{p} I_i^{(j)}(b,\theta_0,w)\left\{\frac{1}{2} - \frac{\nu(t-t_i)}{2\pi}\right\}, \quad (j=1,2)\,.$$

Here $f_0^{(j)}(b)$ is the mean value of the expectation of $f^{(j)}$, and $I_i^{(j)}(b)$ is the expectation of $I_i^{(j)}(b,\theta_o,w)$, the integrals are understood as primitives that have mean value zero. Substituting expressions (5.88) into (5.87) we get a system for first order approximations,

$$\begin{aligned} &\frac{db}{dt} = \frac{\varepsilon}{\mu}F(b), && \frac{d\theta_0}{dt} = \mu - \frac{\varepsilon}{\mu b}\Phi(b), \\ &F(b) = f_0^{(1)}(b) + \frac{\nu}{2\pi}\sum_{i=1}^{p}\bar{I}_i^{(1)}(b), && \Phi(b) = f_0^{(2)}(b) + \frac{\nu}{2\pi}\sum_{i=1}^{p}\bar{I}_i^{(2)}(b)\,. \end{aligned} \tag{5.89}$$

Let $M|\xi(t)|^{6+3\delta} < \infty$ for some $\delta > 0$ and suppose that the complete regularity conditions a)–d) in the preceding section are satisfied with $\beta(\tau) = O(\tau^{-\lambda})$ for $\tau \to \infty$, $\lambda > 16$. Then the condition of Theorem 5.5 are satisfied, and (5.54) gives that

$$A_{11}(a,\varphi) = A_{11}(a)$$

$$= \frac{1}{2\pi\mu^3}\int_0^{2\pi} dt \int_{-\infty}^{\infty} ds K\left(\frac{t-s}{\mu}\right)\psi(a\sin t, a\mu\cos t)\psi(a\sin s, a\mu\cos s)\cos t\cos s;$$

$$A_{22}(a,\varphi) = A_{22}(a)$$

$$= \frac{1}{2\pi a^2\mu^3}\int_0^{2\pi} dt \int_{-\infty}^{\infty} ds K\left(\frac{t-s}{\mu}\right)\psi(a\sin t, a\mu\cos t)\psi(a\sin s, a\mu\cos s)\cos t\cos s;$$

$$A_{12}(a,\varphi) = A_{12}(a)$$

$$= \frac{1}{2\pi a\mu^3}\int_0^{2\pi} dt \int_{-\infty}^{\infty} ds K\left(\frac{t-s}{\mu}\right)\psi(a\sin t, a\mu\cos t)\psi(a\sin s, a\mu\cos s)\cos t\cos s,$$

where $K(t-s)$ is the correlation matrix of the process $\xi(t,w)$ and $\varphi = \mu t + \theta$.

Let $(\rho(\tau, w), r(\tau, w))$ be a random process defined by the system

$$d\rho = \sigma_{11}(b(\tau))d\xi_1(\tau) + \frac{1}{\mu}\frac{dF(b(\tau))}{db}\rho(\tau)d\tau,$$
$$dr = \sigma_{21}(b(\tau))d\xi_1(\tau) + \sigma_{22}(b(\tau))d\xi_2(\tau) + \frac{1}{\mu}\frac{d}{db}\left(\frac{\Phi(b(\tau))}{b(\tau)}\right)\rho(\tau)d\tau,$$

where $\rho(0) = 0$, $r(0) = 0$, $\sigma_{11}(a) = \sqrt{A_{11}(a)}$, $\sigma_{21}(a) = \frac{A_{21}(a)}{\sqrt{A_{11}(a)}}$, $\sigma_{22}(a) = \sqrt{A_{22}(a) - \frac{A_{21}^2(a)}{\sqrt{A_{11}(a)}}}$, $\xi_1(\tau)$, $\xi_1(\tau)$ are Gaussian processes with independent increments, $\tau = \varepsilon t$ is slow time. It then follows from Theorem 5.5 that the process $(\frac{1}{\sqrt{\varepsilon}}(a(\frac{t}{\varepsilon}) - b(t)), \frac{1}{\sqrt{\varepsilon}}(\theta(\frac{t}{\varepsilon}) - \xi(t)))$ converges, as $\varepsilon \to 0$, on the segment $[0, L]$ to the process $(\rho(t), r(t))$ in the sense of convergence of finite dimensional distributions. Here $\theta(t) = \mu t + \varphi$ and $\xi(t) = \mu t + \theta_0$. The process $(\rho(t), r(t))$ is a Markov Gaussian processes that has the probability distribution admitting some explicit representation.

The expression for density assumes an especially simple form if the system starts in a point for which $b(0) = b_0$, where b_0 is a root of the equation $F(b_0) = 0$ (as shown in [105], it is the amplitude of a limit cycle for the system as $\varepsilon \to 0$), and $\frac{d}{da}(\frac{\Phi(b_0)}{b_0}) = 0$ (this condition is satisfied for most real systems). The obtained result permits to substantiate, in a way similar to systems without impulses, the method developed in [16, 17] for investigating systems that appear in radio devices, since it was assumed in the above works, without a sufficient reasoning, that the process in the input of the system is Markov, and then the method of Kolmogorov-Fokker-Planck was applied.

Similar arguments can also be applied to the resonance case.

4. Let us apply the results obtained above to an investigation of linear impulsive differential systems of the form

$$\begin{aligned} &\frac{dx}{dt} = \varepsilon A(t, w)x, \qquad t \neq t_i \\ &\Delta x|_{t=t_i} = \varepsilon B_i(w)x. \end{aligned} \tag{5.90}$$

In this case, solutions of equations (5.33), (5.66) and some other formulas can be written explicitly.

Let us assume that the following conditions hold:

1) The elements of the matrix $A(t, w) - a_{kj}(t, w)$ are periodic in the strict sense, and make periodically coupled random processes;

2) the elements of the matrices $B_i(w) - b_i^{kj}(w)$ are also periodic in the strict sense, with period $p \in \mathbf{N}$, and make periodically coupled random variables, where $t_{i+p} - t_i = \theta$ and t_i are times of the impulsive effects;

3) $\| A(t,w) \| + \| B_i(w) \| < c < \infty$ for arbitrary $t \in \mathbf{R}$, $i \in \mathbf{N}$, and $A(t,w)$ and $B_i(w)$ satisfy conditions a)–d) in Section 2.2.

We also assume that the following conditions are satisfied: condition 1), the first condition in (5.3) in Section 2.1, and condition (5.22) in Section 2.2. It is then easy to see that all conditions of Theorem 5.4 are satisfied. Indeed, condition 1) in Theorem 5.4 is clear. And, since

$$\left| \int_0^\tau \left[\mathbf{E} A\left(\frac{s}{\varepsilon}, w\right) \bar{x}(s) - \bar{A}\bar{x}(s) \right] ds \right| = \varepsilon \left| \int_0^{\frac{\tau}{\varepsilon}} [\mathbf{E} A(s,w) - \bar{A}] \bar{x}(\varepsilon s) ds \right| \to 0\,,$$

as $\varepsilon \to 0$, condition (5.62) is verified. In the same way, one can establish (5.63) and, by constructing the function $\psi(t,x)$ for $\sum\limits_{0<\tau_i<\tau} \mathbf{E} B_i \bar{x}(\tau_i)$ in the same way as in Theorem 5.3, one can similarly show that conditions (5.64) and (5.65) are satisfied. We also have

$$\bar{A} = \frac{1}{\theta} \int_0^\theta \mathbf{E} A(t,w) dt, \bar{B} = \frac{1}{\theta} \sum_{i=1}^p \mathbf{E} B_i(w).$$

Denote $\bar{a}_{kj}^{in} = \frac{1}{\theta} \int\limits_0^\theta ds \int\limits_{-\infty}^{\infty} dt a_{kj}^{in}(s,t)$, where $a_{kj}^{in}(s,t)$ is the correlation tensor of the process $A(t,w)$. In the case under consideration, we have that

$$X_0(x) + I_0(x) = (\bar{A} + \bar{B})x, \qquad A_{kj}(x) = \sum_{j,n} \bar{a}_{kj}^{in} x_i x_j,$$

and so

$$\bar{x}(\tau) = e^{(\bar{A}+\bar{B})\tau} x_0, \qquad X^{(0)}(\tau) = \int_0^\tau e^{(\bar{A}+\bar{B})(\tau - s)} dw(s), \tag{5.91}$$

where $w(s)$ is a Gaussian process with independent increments and the correlation matrix $\int\limits_0^\tau A_{kj}(\bar{x}(s)) ds$.

Let the real parts of all eigen values of the matrix $\bar{A} + \bar{B}$ be negative. Then $\bar{x}(\tau) \to 0$ as $\tau \to \infty$, and $X^{(0)}(\tau)$ tends to zero in probability as $\tau \to$

∞. It is also easy to obtain an approximate representation for solutions of equation (5.90) that would be valid for sufficiently small ε and sufficiently large εt, $x(t) \sim \sqrt{\varepsilon} X^{(0)}(\varepsilon t)$, where $X^{(0)}(\varepsilon t)$ is given by (5.91).

If the matrix $\bar{A} + \bar{B}$ has eigen values with positive real parts, then Theorem 5.4 yields that system (5.90) is not dissipative, that is, all its solutions are unbounded in probability.

Let us now consider the linear nonhomogeneous system

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon(A(t,w)x + f(t,w)), t \neq t_i, \\ \Delta x|_{t=t_i} &= \varepsilon(B_i(w)x + c_i(w)). \end{aligned} \tag{5.92}$$

Let, for example, $f(t,w)$ and $c_i(w)$ do not depend on $A(t,w)$, and $B_i(w)$ satisfy conditions a)–d) in Section 5.2 with the same function $\beta(\tau)$, and $\mathbf{E}f(t,w) = \mathbf{E}c_i(w) = 0$. Assume that all eigen values of the matrix $\bar{A} + \bar{B}$ have negative eigen values. Then it is clear that $\bar{x}(\tau) \to 0$ as $\tau \to \infty$, and $X^{(0)}(\tau)$ is a Markov Gaussian processes satisfying the equation

$$dX^{(0)}(\tau) = dw(\tau) + (\bar{A} + \bar{B})X^{(0)}(\tau)d\tau.$$

This process is stationary. It then follows from Theorem 5.4 that the solution of equation (5.92) has the approximate representation $x(t) \sim \sqrt{\varepsilon} X^{(0)}(\varepsilon t)$ that takes place for sufficiently small ε.

5.4 Averaging for systems with impulsive effects at random times

In the preceding section, the averaging method has been extended to cover impulsive systems in which the impulse times were deterministic. We now will treat the case when both the values of the impulses and the times at which they occur are random.

So, consider a differential system with impulsive effects occurring at random times,

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon X(t,x,\omega), \qquad t \neq t_i(\omega), \\ \Delta x|_{t=t_i(\omega)} &= \varepsilon I_i(x,\omega), \end{aligned} \tag{5.93}$$

where $X(t,x,\omega)$ is a random processes, $I_i(x,\omega), t_i(\omega)$ are random variables, $t \geq 0$, $x \in \mathbf{R}^n, i \in \mathbf{N}$, and ε is a small positive parameter.

We aim at establishing that the solutions of system (5.93) and the solutions of the corresponding averaged deterministic system are mean close.

To this end, let us prove a generalization of the main theorem in [66, Ch. 1]. Consider the differential systems

$$\frac{dx}{dt} = \varepsilon[X_0(t,x) + X_1(t,x,\omega)] = \varepsilon X(t,x,\omega) \tag{5.94}$$

and

$$\frac{d\xi}{dt} = \varepsilon X_0(t,\xi), \tag{5.95}$$

with the condition that $\xi(0) = x(0) = x_0$.

Lemma 5.3. *Let $X(t,x,\omega)$ satisfy the following conditions:*

1) $X(t,x,\omega)$ is a measurable random processes for arbitrary $x \in \mathbf{R}^n$;

2) $X(t,x,\omega)$ is a function continuous with probability 1 in x for arbitrary $t \geq 0$;

3) there exist a random process $m(t,\omega)$, locally integrable on $\mathbf{R}^+$, functions $H(t)$, $m_0(t)$, $H_0(t)$ such that

a)
$$|X(t,x,\omega)| \leq m(t,\omega)\,,$$

b)
$$|X(t,x_1,\omega) - X(t,x_2,\omega)| \leq H(t)|x_1 - x_2|\,,$$

c)
$$|X_0(t,x_1) - X_0(t,x_2)| \leq H_0(t)|x_1 - x_2|,\ |X_0(t,x)| \leq m_0(t)\,,$$

for arbitrary x, x_1, $x_2 \in \mathbf{R}^n$, $t \geq 0$;

4) there exists a constant $C > 0$ such that for arbitrary $T > 0$,

$$\int_0^T [H(t) + H_0(t) + m_0(t)]\,dt \leq CT;$$

5) the limit

$$\lim_{T\to\infty} \mathbf{E}\frac{1}{T}\left|\int_0^T X_1(t,x,\omega)\,dt\right| = 0$$

exists uniformly in $x \in \mathbf{R}^n$.

Then, for arbitrary $\eta > 0$ and $L > 0$ there exists $\varepsilon_0 > 0$ such that

$$\mathbf{E}|x(t) - \xi(t)| \le \eta\,, \tag{5.96}$$

if $t \in [0,\ \frac{L}{\varepsilon}]$ for all $\varepsilon < \varepsilon_0$.

Proof. Let us represent (5.94) and (5.95) in an integral form,

$$x(t) = x_0 + \varepsilon \int_0^t X(s, x(s), \omega)\, ds\,,$$

$$\xi(t) = x_0 + \varepsilon \int_0^t X_0(s, \xi(s))\, ds\,.$$

Then

$$\begin{aligned}|x(t) - \xi(t)| &\le \varepsilon \int_0^t |X(s, x(s), \omega) - X(s, \xi(s), \omega)|\, ds \\ &\quad + \varepsilon \left| \int_0^t [X(s, \xi(s), \omega) - X_0(s, \xi(s)]\, ds \right| \le \varepsilon \int_0^t H(s)|x(s) - \xi(s)|\, ds \\ &\quad + \varepsilon \left| \int_0^t X_1(s, \xi(s), \omega)\, ds \right|,\end{aligned}$$

and, hence,

$$\mathbf{E}|x(t) - \xi(t)| \le \varepsilon \int_0^t H(s)\mathbf{E}|x(s) - \xi(s)|\, ds + \varepsilon \mathbf{E}\left| \int_0^t X_1(s, \xi(s), \omega)\, ds \right|. \tag{5.97}$$

Let us estimate the second term in (5.97). It easily follows from condition 4) that there is a partition of the segment $[0,\ \frac{L}{\varepsilon}]$ with points $\{t_i(\varepsilon)\}_{i=0}^{n-1}$ such that the following estimate holds on an arbitrary segment $[t_i,\ t_{i+1}]$:

$$\int_{t_i}^{t_{i+1}} m_0(t)\, dt \le \frac{CL}{n}.$$

Then we have that

$$\begin{aligned}\varepsilon \mathbf{E}\left| \int_0^{\frac{L}{\varepsilon}} X_1(s, \xi(s), \omega)\, ds \right| &= \Bigg| \sum_{i=0}^{n-1} \varepsilon \mathbf{E} \int_{t_i}^{t_{i+1}} [X_1(t, \xi(t), \omega) \\ &\quad - X_1(t, \xi(t_i), \omega)]\, dt + \sum_{i=0}^{n-1} \varepsilon \mathbf{E} \int_{t_i}^{t_{i+1}} X_1(t, \xi(t_i), \omega)\, dt \Bigg|.\end{aligned} \tag{5.98}$$

However, on the interval $[t_i,\ t_{i+1}]$, we have

$$|\xi(t) - \xi(t_i)| \le \int_{t_i}^{t_{i+1}} |X_0(t, \xi(t))|\, dt \le \int_{t_i}^{t_{i+1}} m_0(t)\, dt \le \frac{CL}{n},$$

which together with conditions a)–c) give

$$\begin{aligned}
&\sum_{i=0}^{n-1} \varepsilon \mathbf{E} \int_{t_i}^{t_{i+1}} |X_1(t, \xi(t), \omega) - X_1(t, \xi(t_i), \omega)|\, dt \\
&\quad \le \sum_{i=0}^{n-1} \varepsilon \int_{t_i}^{t_{i+1}} H_1(t) |\xi(t) - \xi(t_i)|\, dt \\
&\quad \le \sum_{i=0}^{n-1} \varepsilon \int_{t_i}^{t_{i+1}} H_1(t)\, dt \frac{CL}{n} \le \frac{CL}{n} \varepsilon \int_0^{\frac{L}{\varepsilon}} H_1(t)\, dt \le \frac{C^2 L^2}{n}\,. \qquad (5.99)
\end{aligned}$$

(Existence of $H_1(t)$ with property 4) follows from conditions b), c), and 4) of the lemma).

Let us now fix n such that

$$\frac{C^2 L^2}{n} < \frac{\eta}{2} \exp\{-CL\}\,. \qquad (5.100)$$

By condition 5) of the lemma, there exists a monotone decreasing function $f(t) \to 0$, $t \to \infty$, such that

$$\mathbf{E} \left| \int_0^T X_1(t, x, \omega)\, dt \right| \le T f(T)\,. \qquad (5.101)$$

Inequality (5.101) permits, as in [66], to estimate the second sum in (5.98) and to obtain the estimate

$$\sum_{i=0}^{n-1} \varepsilon \mathbf{E} \left| \int_{t_1}^{t_{i+1}} X_1(t, \xi(t_i), \omega) \right| dt \le \frac{\eta}{2} \exp\{-CL\}\,. \qquad (5.102)$$

It follows from (5.97), (5.98), (5.101), (5.102), and the Gronwall–Bellman lemma that estimate (5.96) holds. □

The preceding lemma permits to substantiate the averaging method for impulsive systems of form (5.93).

Theorem 5.7. *Let the following conditions hold:*

1) $X(t, x, \omega)$ is a measurable random process;

2) $X(t,x,\omega)$ is continuous and $I_i(x,\omega)$ is continuously differentiable in $x \in \mathbf{R}^n$ with probability 1;

3) there exists a constant $C > 0$ such that

$$\mathbf{E}\left(\sup_{T>0}\frac{1}{T}\int_0^T \sup_{x\in\mathbf{R}^n}|X(t,x,\omega)|\,dt\right)^2 \le C, \qquad \mathbf{E}\left(\sup_{i\in\mathbf{N},x\in\mathbf{R}^n}|I_i(x,\omega)|^2\right) \le C\,;$$

4) there exists a locally integrable function $H(t)$, a sequence of random variables l_i, and a number $l > 0$ such that

a)

$$|X(t,x_1,\omega) - X(t,x_2,\omega)| \le H(t)|x_1 - x_2|\,,$$

b)

$$|I_i(x_1,\omega) - I_i(x_2,\omega)| \le l_i|x_1 - x_2|$$

with probability 1,

and

$$\sup_{T>0}\frac{1}{T}\int_0^T H(t)\,dt + \mathbf{E}\left(\sup_{T>0}\frac{1}{T}\sum_{0<t_i<T} l_i\right) \le C, \qquad l_i \le l\,,$$

for arbitrary $x_1, x_2 \in \mathbf{R}^n$, $t \ge 0$, $i \in \mathbf{N}$, where C is the constant that enters in condition 3);

5)

$$\mathbf{E}\sup_{T>0}\left(\frac{1}{T}\sum_{0<t_i<T}\sup_{x\in\mathbf{R}^n}|I_i(x,\omega)|\right)^2 \le C\,;$$

6) the sequence $t_i(\omega)$ does not have finite accumulation points with probability 1;

7) the limit

$$\lim_{T\to\infty}\frac{1}{T}\mathbf{E}\left|\int_0^T X(t,x,\omega)\,dt + \sum_{0<t_i<T} I_i(x,\omega) - X_0(x)\right| = 0$$

exists uniformly in $x \in \mathbf{R}^n$.

Then, for arbitrary $\eta > 0$ *there exists* $\varepsilon_0 > 0$ *such that*

$$\mathbf{E}|x(t) - \xi(\varepsilon t)| < \eta$$

for arbitrary $\varepsilon < \varepsilon_0,\ L > 0$ *and every* $t \in [0,\ \frac{L}{\varepsilon}]$, *where* $\xi = \xi(\varepsilon t)$ $(\xi(0) = x_0)$ *is a solution, defined for* $\tau = \varepsilon t \in [0,\ L]$, *of the averaged deterministic system*

$$\frac{d\xi}{d\tau} = X_0(\xi)\,.$$

Proof. Without loss of generality, we can assume that $L = 1$.

By apply the change of variables suggested in [184], we make a piecewise smooth change of variables

$$x = y + \varepsilon\frac{t_{i+1} - t}{t_{i+1} - t_i} I_i(y, \omega) \tag{5.103}$$

in system (5.93) on every interval $(t_i, t_{i+1}]$. It is easy to see that the impulsive system (5.93) is reduced to an ordinary differential system with random right-hand side,

$$\frac{dy}{dt} = \varepsilon\left(X(t, y, \omega) + \frac{I_i(y, \omega)}{t_{i+1} - t_i}\right) + Q(t, y, \varepsilon)\,, \tag{5.104}$$

for $t \in (t_i,\ t_{i+1}]$ and sufficiently small ε. Here, $Q(t, y, \varepsilon)$ verifies the following estimate with probability 1:

$$\begin{aligned}|Q(t, y, \varepsilon)| \le \varepsilon^2 H(t)|I_i(y, \omega)| + \varepsilon^2\frac{l}{\delta}\left[|X(t, y, \omega)| + \frac{|I_i(y, \omega)|}{t_{i+1} - t_i}\right]\\ + \varepsilon^3 H(t)|I_i(y, \omega)|\frac{l}{\delta}\,,\end{aligned} \tag{5.105}$$

where $\delta > 0$, and ε is chosen as to satisfy $1 - \varepsilon l > \delta > 0$.

Consider he system

$$\frac{dz}{dt} = \varepsilon\left(X(t, z, \omega) + \frac{I_i(z, \omega)}{t_{i+1} - t_i}\right) \tag{5.106}$$

for $t \in (t_i,\ t_{i+1}]$ obtained from (5.104) by dropping the term $Q(t, y, \varepsilon)$ that is of order ε^2 as compared with other terms.

Estimate the difference $\mathbf{E}|y(t) - z(t)|$ between the corresponding solutions of systems (5.104) and (5.106). To this end, represent the solutions in an integral form,

$$y(t) = y_0 + \int_0^t \left(\varepsilon\left[X(s, y(s), \omega) + \frac{I_i(y(s), \omega)}{t_{i+1} - t_i}\right] + Q(s, y(s), \omega)\right) ds\,,$$

$$z(t) = y_0 + \int_0^t \varepsilon \left[X(s, z(s), \omega) + \frac{I_i(z(s), \omega)}{t_{i+1} - t_i} \right] ds\,,$$

where I_i is chosen depending on s that belongs to the interval $(t_i,\ t_{i+1}]$.

Consider auxiliary systems constructed as follows. Let

$$A_N = \left\{ \omega : \sup_{\varepsilon>0} \varepsilon \sum_{0<t_i<\frac{1}{\varepsilon}} l_i \le N \right\}.$$

Set

$$\frac{dy_N}{dt} = \varepsilon \left(X(t, y_N, \omega) + \frac{I_i(y_N, \omega)}{t_{i+1} - t_i} \chi_N(\omega) \right) + Q(t, y_N, \varepsilon) \tag{5.107}$$

and

$$\frac{dz_N}{dt} = \varepsilon \left(X(t, z_N, \omega) + \frac{I_i(z_N, \omega)}{t_{i+1} - t_i} \chi_N(\omega) \right), \tag{5.108}$$

where $\chi_N(\omega)$ is the characteristic function of the set A_N.

Let us show that $\mathbf{E}|y(t) - y_N(t)|$ and $\mathbf{E}|z(t) - z_N(t)|$ tend to zero uniformly in ε and $t \in [0,\ \frac{1}{\varepsilon}]$ as $N \to \infty$. Indeed, for arbitrary $N_1 < N_2 < \infty$, by condition 4) of the theorem,

$$\begin{aligned} &\mathbf{P} \left\{ \omega : \sup_{\varepsilon>0} \sup_{t\in[0,\ \frac{1}{\varepsilon}]} |y_{N_1}(t) - y_{N_2}(t)| > 0 \right\} \\ &\qquad \le \mathbf{P} \left\{ \omega : \sup_{\varepsilon>0} \varepsilon \sum_{0<t_i<\frac{1}{\varepsilon}} l_i > N_1 \right\} \le \frac{M(\sup_{\varepsilon>0} \varepsilon \sum_{0<t_i<\frac{1}{\varepsilon}} l_i)}{N_1} \to 0\,, \end{aligned}$$

as $N_1 \to \infty$. This means that the sequence $y_N(t)$ is Cauchy in probability uniformly in $\varepsilon > 0$ and $t \in [0,\ \frac{1}{\varepsilon}]$. On the other hand, it is easy to see that $y_N(t) \to y(t)$.

In the same way, one proves that the convergence of the sequence $z_N(t) \to z(t)$ in probability as $N \to \infty$ is uniform in $\varepsilon,\ t$.

Let us show that an appropriate choice of ε_0 makes the difference $\mathbf{E}|y_N(t) - z_N(t)|$ arbitrarily small for arbitrary N and $t \in [0,\ \frac{1}{\varepsilon}]$ as $\varepsilon \in (0,\ \varepsilon_0]$.

Indeed, the integral representation of $y_N(t)$ and $z_N(t)$, with a use of condition 4) and the Gronwall–Bellman lemma, gives

$$\begin{aligned} |y_N(t) - z_N(t)| &\le \int_0^{\frac{1}{\varepsilon}} |Q(t, y_N(t), \varepsilon)|\, dt \exp \left\{ \varepsilon \int_0^{\frac{1}{\varepsilon}} H(t)\, dt + \varepsilon \sum_{0<t_i<\frac{1}{\varepsilon}} l_i \chi_N(\omega) \right\} \\ &\le \exp\{C + N\} \int_0^{\frac{1}{\varepsilon}} \mathbf{E}|Q(t, y_N(t), \varepsilon)|\, dt\,. \end{aligned}$$

Hence, $\mathbf{E}|y_N(t)-z_N(t)| \le \exp\{C+N\}\int_0^{\frac{1}{\varepsilon}} \mathbf{E}|Q(t,y_N(t),\varepsilon)|\,dt$. So, using now conditions 3) and 5) of the theorem we get the estimate

$$\int_0^{\frac{1}{\varepsilon}} \mathbf{E}|Q(t,y_N(t),\varepsilon)|\,dt \le \varepsilon B,$$

where the constant B does not depend on ε and N. And, hence,

$$\mathbf{E}|y_N - z_N| \le \varepsilon B \exp\{C+N\},$$

for arbitrary $t \in [0,\ \frac{1}{\varepsilon}]$.

It immediately follows from the conditions of the theorem that

$$\mathbf{E} \sup_{\varepsilon>0, t\in[0\ \frac{1}{\varepsilon}]} |y(t)-y_N(t)|^2 < C$$

and

$$\mathbf{E} \sup_{\varepsilon>0, t\in[0\ \frac{1}{\varepsilon}]} |z(t)-z_N(t)|^2 < C.$$

This allows to take the limit for $N \to \inf ty$ under the expectation uniformly in $\varepsilon > 0$ and $t \in [0,\ \frac{1}{\varepsilon}]$. So, on the interval $[0,\ \frac{1}{\varepsilon}]$, we have

$$\mathbf{E}|y(t)-z(t)| \le \mathbf{E}|y(t)-y_N(t)| + \mathbf{E}|y_N(t)-z_N(t)| + \mathbf{E}|z_N(t)-z(t)|.$$

Choose now N so large that $\mathbf{E}|y(t)-y_N(t)|$ and $\mathbf{E}|z_N(t)-z(t)|$ would be less than $\frac{\eta}{9}$, and ε so small that the inequality

$$\mathbf{E}|y_N(t)-z_N(t)| \le \frac{\eta}{9}$$

would hold. Then $\mathbf{E}|y(t)-z(t)| \le \frac{\eta}{3}$ for arbitrary $\varepsilon < \varepsilon_0,\ t \in [0,\ \frac{1}{\varepsilon}]$.

However, by the conditions of the theorem, the preceding lemma can be applied to the system, showing that for an arbitrary $\varepsilon > 0$ there exists $\varepsilon_1 > 0$ such that the following estimate holds for arbitrary $\varepsilon < \varepsilon_1$:

$$\mathbf{E}|\xi(t)-z(t)| < \frac{\eta}{3}, \qquad \forall t \in \left[0,\ \frac{1}{\varepsilon}\right].$$

Since

$$x(t) = y(t) + \varepsilon\frac{t_{i+1}-t}{t_{i+1}-t_i} I_i(y(t),\omega)$$

for $\varepsilon < \min\{\varepsilon_0,\ \varepsilon_1\}$, we have

$$\mathbf{E}|x(t)-\xi(t)| \le \mathbf{E}|y(t)-z(t)| + \varepsilon\mathbf{E}\sup_{y\in\mathbf{R}^n}|I_i(y,\omega)|$$

$$+\ \mathbf{E}|z(t)-\xi(t)| \le \frac{\eta}{3} + \varepsilon C^{\frac{1}{2}} + \frac{\eta}{3} < \eta$$

for arbitrary $\varepsilon < \varepsilon_2$ chosen so that $\varepsilon_2 C^{\frac{1}{2}} < \frac{\eta}{3}$, $\varepsilon_2 < \min\{\varepsilon_0,\ \varepsilon_1\}$. $\square$

Remark. Conditions of Theorem 5.7 can be somewhat weakened if we assume that the times of the impulsive effects are jointly independent. Indeed, denote by $\tau_i = t_{i+1} - t_i$ the random variables that give the time between two consecutive times of impulses, $t_0 = 0$. It is natural to regard them as independent and identically distributed. In such a case, if $M\tau_i > 0$, condition 6) of Theorem 5.7 is always satisfied.

Indeed, let us show that $t_i \to \infty$, $i \to \infty$, with probability 1. We have

$$t_i = \tau_0 + \tau_1 + \cdots + \tau_{i-1}\,.$$

Then, for arbitrary $C > 0$,

$$\begin{aligned}\mathbf{P}\{t_i \le C\} &= \mathbf{P}\left\{\exp\left\{-\sum_{k=0}^{i}\tau_k\right\} \ge \exp\{-C\}\right\}\\ &\le \exp\{C\}\mathbf{E}\exp\{-\tau_1\}\ldots\exp\{-\tau_{i-1}\} = \exp\{C\}a^i\,,\end{aligned}$$

where $a = \mathbf{E}\exp\{-\tau_0\}$. Hence,

$$\mathbf{P}\{t_i \le C\} \to 0, \qquad i \to \infty\,,$$

and, since

$$\{t_i \le C\} \supset \{t_{i+1} \le C\},$$

we see that

$$\mathbf{P}\left\{\lim_{i\to\infty} t_i \le C\right\} = \lim_{i\to\infty}\mathbf{P}\{t_i \le C\} = 0,$$

which shows that condition 6) of Theorem 5.7 is satisfied.

5.5 The second theorem of M. M. Bogolyubov for systems with regular random perturbations

We will consider a differential system with random right-hand side and a small parameter,

$$\frac{dx}{dt} = \varepsilon X_1(t,x) + \varepsilon^2 X_2(t,x,\xi(t))\,, \tag{5.109}$$

where $X_1(t,x)$ and $X_2(t,x,y)$ are functions defined and jointly continuous on $\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^m$, periodic in t with period θ, $\xi(t)$ is a random process that is periodic in the narrow sense, having continuous trajectories, and taking values in the space $\mathbf{R}^m$.

For such systems, we prove an analogue of the second theorem of M. M. Bogolyubov giving a substantiation of an averaging method, namely, we investigate the correspondence between solutions of system (5.109) and the equilibrium positions of the averaged system.

Let
$$X_0(x) = \frac{1}{\theta}\int_0^\theta X_1(t,x)\,dt\,.$$
Together with system (5.109), consider the deterministic system of averaged equations,
$$\frac{dx}{dt} = \varepsilon X_0(x)\,. \tag{5.110}$$
Let $x = x_0$ be an isolated equilibrium position of system (5.110). Denote
$$B(t,x) = \int_0^t [X_1(s,x) - X_0(x)]\,ds.$$

Theorem 5.8. *Let system* (5.109) *satisfy the following conditions:*

1) $X_1(t,x)$ and $X_2(t,x,z)$ are Lipschitz continuous in x for all t, z in the definition domain;

2) there exists a constant $C > 0$ such that $|X_2(t,x_0,z)| \le C$ for arbitrary t and z;

3) the function $X_1(t,x)$ is twice continuously differentiable in x in some ρ-neighborhood of the point x_0, and $X_2(t,x,z)$ is continuously differentiable in x;

4) all real parts of the eigen values of the matrix
$$H = \frac{\partial X_0(x_0)}{\partial x}$$
are nonzero.

Then there exists ε_0 such that for every $\varepsilon < \varepsilon_0$ equation (5.109) *has a solution that is periodic with period θ in the sense of finite dimensional distributions and periodically connected with $\xi(t)$.*

Moreover, if the real parts of the eigen values of the matrix H are negative, then there is a θ-periodic solution $x(t,\varepsilon)$ of system (5.109) *in a neighborhood of the point x_0 such that*

1)

$$\lim_{\varepsilon\to 0} x(t,\varepsilon) = x_0$$

uniformly in $t \in \mathbf{R}$ with probability 1*;*

2) $x(t,\varepsilon)$ is asymptotically stable with probability 1 *and exponentially decaying as $t \to \infty$.*

Proof. Let us make the change of variables

$$x = y + \varepsilon B(t,y) \tag{5.111}$$

in system (5.109). One can find $\rho_1 < \rho$ such that if $|y - x_0| \le \rho_1$, the point x will be in a ρ-neighborhood of x_0 for sufficiently small ε.

We have

$$\begin{aligned}\frac{dx}{dt} &= \frac{dy}{dt} + \varepsilon\frac{\partial B}{\partial t} + \varepsilon\frac{\partial B}{\partial y}\frac{dy}{dt} = \left[E + \varepsilon\frac{\partial B}{\partial y}\right]\frac{dy}{dt} \\ &= \varepsilon X_0(y) + \varepsilon X_1(t, y + \varepsilon B(t,y)) \\ &\quad - \varepsilon X_1(t,y) + \varepsilon^2 X_2(t, y + \varepsilon B(t,y), \xi(t))\,. \end{aligned} \tag{5.112}$$

It is easy to see that the matrix

$$E + \varepsilon\frac{\partial B}{\partial y}$$

has an inverse for sufficiently small ε and, hence, (5.112) can be explicitly solved with respect to $\frac{dy}{dt}$, obtaining the system

$$\frac{dy}{dt} = \varepsilon X_0(y) + \varepsilon X_1(t, y + \varepsilon B(t,y)) - \varepsilon X_1(t,y) + \varepsilon^2 R(t,y,\omega)\,, \tag{5.113}$$

where, by the conditions of the theorem, $R(t,x,\omega)$ is a random function that, together with its partial derivatives with respect to y, is bounded with probability 1 in a ρ-neighborhood of the point x_0 with some constant.

By passing to a variable z in (5.113) by the formula

$$y = x_0 + z,$$

we obtain the following system in the ρ_1-neighborhood of the point x_0:

$$\frac{dz}{dt} = \varepsilon\frac{\partial X_0(x_0)}{\partial y}z + \left\{\varepsilon\left[X_0(x_0 + z) - X_0(x_0) - \frac{\partial X_0(x_0)}{\partial y}z\right]\right.$$

$$+\ \varepsilon[X_1(t, x_0 + z + \varepsilon B(t, x_0 + z)) - X_1(t, x_0 + z)]\Big\} + \varepsilon^2 R(t, z, \omega)\,. \tag{5.114}$$

Using the conditions of the theorem, we have

$$\left\|\frac{\partial X_0(x_0 + z)}{\partial y} - \frac{\partial X_0(x_0)}{\partial y}\right\| \le r(z) \to 0, \qquad |z| \to 0\,,$$
$$|X_1(t, x_0 + \varepsilon B(t, x_0)) - X_1(t, x_0)| \le L\varepsilon B(t, x_0) \to 0, \qquad \varepsilon \to 0\,.$$

For the partial derivatives, we have

$$\left\|\frac{\partial X_1(t, x_0 + z + \varepsilon B(t, x_0 + z))}{\partial z}\left(E + \varepsilon\frac{\partial B(t, x_0 + z)}{\partial z}\right) - \frac{\partial X_1(t, x_0 + z)}{\partial z}\right\|$$
$$\le L\varepsilon B(t, x_0 + z) + \varepsilon C_1 \to 0, \qquad \varepsilon \to 0\,,$$

which show that the partial derivative with respect to z of the function in parentheses in (5.114) can not exceed some value $\lambda(\varepsilon\,\sigma) \to 0,\ \sigma \to 0$, for $|z| \le \sigma < \rho_1$. This allows to write system (5.114) as

$$\frac{dz}{dt} = \varepsilon Hz + \varepsilon\Phi(t, z, \omega, \varepsilon)\,. \tag{5.115}$$

Using the "slow" time $\tau = \varepsilon t$ in (5.115) and again replacing τ with t we get the system

$$\frac{dz}{dt} = Hz + Q(t, z, \omega, \varepsilon)\,, \tag{5.116}$$

where $Q(t, z, \omega, \varepsilon) = \Phi(\frac{t}{\varepsilon}, z, \omega, \varepsilon)$.

It is clear that the function $Q(t, z, \omega, \varepsilon)$ has the following properties:

1) $Q(t, z, \omega, \varepsilon)$ is defined on the domain $t \in \mathbf{R}$ for $|z| \le \rho_1$ and sufficiently small ε;

2) $\sup_{t\in\mathbf{R}} |Q(t, z, \omega, \varepsilon)| \le M(\varepsilon)$ with probability 1, where $M(\varepsilon) \to 0$ as $\varepsilon \to 0$;

3) $|Q(t, z, \omega, \varepsilon) - Q(t, z', \omega, \varepsilon)| \le \lambda(\varepsilon, \sigma)|z - z'|$ with probability 1 for arbitrary z and z' in the space $\mathbf{R^m}$.

Let us introduce, as in [103], Green's function $J(t)$ for the linear part of system (5.116). By the conditions of the theorem, there exist positive constants $K > 0,\ \alpha > 0$ such that

$$||J(t)|| \le Ke^{-\alpha|t|}, \qquad t \in \mathbf{R}\,.$$

Fix a positive number $d \le \rho_1$ and consider the class of random processes $\zeta(t)$ that are continuous with probability 1, defined on $\mathbf{R}$ and taking values in $\mathbf{R}^n$, and such that the following inequality holds with probability 1:

$$\sup_{t\in\mathbf{R}} |\zeta(t)| \le d\,. \tag{5.117}$$

Denote this class of processes by $C(d)$. We will solve the integral equation

$$F(t) = \int_{-\infty}^{\infty} J(z)Q(t+z, F(t+z), \omega, \varepsilon)\, dz\,. \tag{5.118}$$

Consider the operator

$$S_t(F) = \int_{-\infty}^{\infty} J(z)Q(t+z, F(t+z), \omega, \varepsilon)\, dz$$

on the class $C(d)$.

Using properties of the function Q we then have

$$\begin{aligned}|Q(t+z, F(t+z), \omega, \varepsilon)| \le |Q(t+z, 0, \omega, \varepsilon)| + |Q(t+z, F(t+z), \omega, \varepsilon)\\ - Q(t+z, 0, \omega, \varepsilon)| \le M(\varepsilon) + \lambda(\varepsilon, d)d\end{aligned}$$

with probability 1.

Hence,

$$\begin{aligned}\sup_{t\in\mathbf{R}} |S_t(F)| &\le \{M(\varepsilon) + \lambda(\varepsilon, d)d\} \int_{-\infty}^{\infty} Ke^{-\alpha|z|}\, dz\\ &= \frac{2K}{\alpha}\{M(\varepsilon) + \lambda(\varepsilon, d)d\}\,. \end{aligned} \tag{5.119}$$

with probability 1. For the two $C(d)$-processes, we also have the estimate

$$\begin{aligned}|S_t(\overline{F}) - S_t(F)| = \bigg| \int_{-\infty}^{\infty} J(z)\{Q(t+z, \overline{F}(t+z), \omega, \varepsilon)\\ - Q(t+z, F(t+z), \omega, \varepsilon)\}\, dz \bigg|\end{aligned}$$

$$\leq \lambda(\varepsilon, d) \int_{-\infty}^{\infty} K e^{-\alpha|z|} |\overline{F}(t+z) - F(t+z)|\, dz$$
$$\leq \frac{2K\lambda(\varepsilon, d)}{\alpha} \sup_{t\in\mathbf{R}} |\overline{F}(t) - F(t)|\,.$$

Choose d, as a function of the parameter ε, such that $d(\varepsilon) \to 0$ as $\varepsilon \to 0$ and such that the following inequality would hold for sufficiently small ε:

$$\frac{2K}{\alpha}\{M(\varepsilon) + \lambda(\varepsilon, d)d\} \leq d, \tag{5.120}$$

$$\frac{4\lambda(\varepsilon, d)}{\alpha} K \leq 1. \tag{5.121}$$

Such $d = d(\varepsilon)$ can be found, since $M(\varepsilon) \to 0$ and $\lambda(\varepsilon, d) \to 0$ as $\varepsilon \to 0$. Then, with probability 1, we have

$$\sup_{t\in\mathbf{R}} |S_t(F)| \leq d(\varepsilon), \tag{5.122}$$

$$\sup_{t\in\mathbf{R}} |S_t(\overline{F} - S_t(F)| \leq \frac{1}{2} \sup_{t\in\mathbf{R}} |\overline{F}(t) - F(t)|. \tag{5.123}$$

Let us solve equation (5.118) by using successive approximation method. Let

$$F_0 = 0, \quad F_1 = S_t(F_0),\ \ldots,\ F_{n+1} = S_t(F_n)\,. \tag{5.124}$$

It follows from (5.122) that all members of the sequence belong to the class $C(d)$, and (5.123) gives

$$\sup_{t\in\mathbf{R}} |F_{n+1}(t) - F_n(t)| \leq \left(\frac{1}{2}\right)^n,$$

which shows that the series

$$F_0(t) + \sum_{n=0}^{\infty} [F_{n+1}(t) - F_n(t)]$$

converges with probability 1 for $t \in \mathbf{R}$. Its sum is a uniform limit of $F_n(t)$ with probability 1 and, hence, $F_n(t)$ converges to some random processes $F(t)$ that is of the class $C(d)$. By passing to the limit in (5.124) we see that $F(t)$ is a solution of equation (5.118). Uniqueness of this solution in the class $C(d)$ follows from estimate (5.123).

Differentiating (5.118) and using properties of Green's function we see that $F(t)$ is a solution of equation (5.116). Then system (5.109) also has a solution $x(t, \varepsilon)$ that satisfies

$$\sup_{t\in\mathbf{R}} |x(t, \varepsilon) - x_0| \leq \rho$$

with probability 1 and, by [70], this implies that system (5.109) has a periodic solution that is periodically connected with $\xi(t)$.

Let now the eigen values of the matrix H have negative real parts. We say that an arbitrary solution of equation (5.116) is of type S if it satisfies the following condition: if $z(t_0) = z_0$ for some $t = t_0$ and $|z_0| \le \rho_1$, then $|z(t)| \le \rho_2$ for arbitrary $t > t_0$, $\rho_1 \le \rho_2 < \rho$. Then, arbitrary solution of type S, $z(t)$, $f(t)$, by [103], satisfies the estimate

$$|f(t) - z(t)| \le K_1 e^{-\alpha(t-t_0)} |f(t_0) - z(t_0)|, \qquad \forall t \ge t_0$$

with probability 1. However, if all eigen values of the matrix H have negative real values, the entire ρ_1-neighborhood of $f(t_0)$, where $f(t)$ is the sought S-type solution, consists of initial values of S-type solutions for all t_0. This gives the second claim of the theorem, since, by [70], the initial value of a periodic solution belongs to the ρ_1-neighborhood of $f(t_0)$. □

Corollary 5.1. *If X_1 and X_2 in system* (5.109) *do not depend on t and $\xi(t)$ is a stationary process, then the conditions of the theorem imply existence of a stationary solution connected with $\xi(t)$ having the same properties as the periodic solution in Theorem 5.8.*

To illustrate Theorem 5.8, consider an example of an ordinary harmonic oscillator influenced by small random perturbations and given by the equation

$$x'' + \mu^2 x = \varepsilon\varphi(\nu t, x, x', \varepsilon, \omega) = \varepsilon f(\nu t, x, x') + \varepsilon^2 f_1(\nu t, x, x', \xi(\nu t))\,, \quad (5.125)$$

where $f,\ f_1$ are 2π-periodic functions with respect to νt and $\xi(\nu t)$ is a random process 2π-periodic in νt, $\mu^2 = (\frac{p\nu}{q})^2 + \varepsilon\Delta$, p and q are mutually prime numbers (the resonance case).

By making the change of variables in (5.125),

$$x = \zeta\cos\left(\frac{p}{q}\nu t\right) + \eta\sin\left(\frac{p}{q}\nu t\right), \quad x' = -\zeta\frac{p}{q}\nu\sin\left(\frac{p}{q}\nu t\right) + \eta\frac{p}{q}\nu\cos\left(\frac{p}{q}\nu t\right), \quad (5.126)$$

we obtain equations in a standard form,

$$\begin{aligned} \zeta' &= \varepsilon X_1(t,\zeta,\eta) + \varepsilon^2 X_2(t,\zeta,\eta,z(t))\,, \\ \eta' &= \varepsilon Y_1(t,\zeta,\eta) + \varepsilon^2 Y_2(t,\zeta,\eta,z(t))\,, \end{aligned} \quad (5.127)$$

where $X_1,\ Y_1,\ X_2,\ Y_2$ are functions 2π-periodic in νt.

The averaged equations corresponding to (5.127) are

$$\zeta' = \varepsilon X_0(\zeta,\eta), \qquad \eta' = \varepsilon Y_0(\zeta,\eta)\,, \tag{5.128}$$

where

$$X_0(\zeta,\eta) = \frac{\nu}{2\pi q}\int_0^{\frac{2\pi q}{\nu}} X_1(t,\zeta,\eta)\,dt, \qquad Y_0(\zeta,\eta) = \frac{\nu}{2\pi q}\int_0^{\frac{2\pi q}{\nu}} Y_1(t,\zeta,\eta)\,dt\,.$$

Assume that the system

$$\begin{cases} X_0(\zeta,\eta) = 0\,, \\ Y_0(\zeta,\eta) = 0\,, \end{cases}$$

has a nonzero solution

$$\zeta = \zeta_0, \qquad \eta = \eta_0 \tag{5.129}$$

and, in a neighborhood of the ellipse

$$x^2 + \frac{(x')^2}{(\frac{p}{q}\nu)^2} = a_0^2\,, \tag{5.130}$$

where $a_0^2 = \zeta_0^2 + \eta_0^2$, the functions f and f_1 are twice continuously differentiable in all its variables, and that the function $f_1(\nu t, x, x', z)$ is bounded on ellipse (5.130) by some constant.

Let the real parts of the eigen values of the matrix

$$\begin{pmatrix} X'_{0\zeta}(\zeta_0,\eta_0) & X'_{0\eta}(\zeta_0,\eta_0) \\ Y'_{0\zeta}(\zeta_0,\eta_0) & Y'_{0\eta}(\zeta_0,\eta_0) \end{pmatrix}$$

be negative. Then all the conditions of Theorem 5.8 are satisfied and, hence, for sufficiently small ε, equation (5.125) has a solution that is $\frac{2\pi q}{\nu}$-periodic, periodically connected with $\xi(\nu t)$, and is close with probability 1 to the harmonic solution

$$x = a_0 \cos\left(\frac{(p\nu t)}{q} + \varphi_0\right),$$

where $a_0 = \sqrt{\zeta_0^2 + \eta_0^2}$, $\varphi_0 = -\arctan(\frac{\zeta_0}{\eta_0})$.

Let us make a remark on the theorem proved in this section. If we restrict the class of random processes $\xi(t)$, then it is possible to obtain more substantial results towards the second theorem of M. M. Bogolyubov.

Let us give without a proof an already announced fine result of V. S. Korolyuk contained in [80] where an investigation of stability of a dynamical system under effects of rapid Markov switchings can be reduced to a study of the averaged deterministic system.

A dynamical system with rapid Markov switchings is given by a system of differential evolution equations

$$\frac{dU^\varepsilon(t)}{dt} = C(U^\varepsilon(t), X^\varepsilon(t)), \qquad U^\varepsilon(0) = u_0\,, \tag{5.131}$$

in $\mathbf{R^d}$. Rapid Markov switchings are generated by a homogeneous, jump-like, uniformly ergodic, Markov process $X^\varepsilon(t) := X(\frac{t}{\varepsilon})$ in a measure space $(X,\, \pounds)$. A corresponding generating operator is defined by the kernel $Q(x, A)$, $x \in X$, $A \in \pounds$, $q(x) := Q(x, X)$.

The stationary distribution $\pi(dx)$ is defined by the relation

$$\int_A q(x)\pi(dx) = \int_X \pi(dy)Q(y, A), \qquad A \in \pounds.$$

Denote by R_0 the corresponding potential operator.

The averaged system is defined by the deterministic evolution equation

$$\frac{dU(t)}{dt} = C(U(t)), \qquad U(0) = u_0\,, \tag{5.132}$$

where the speed of the evolution is given by the relation

$$C(u) := \int_X \pi(dx)C(u, x)\,.$$

Let us also introduce the velocity matrix

$$C^0(u, x) := C(u, x)R_0C^*(u, x)$$

and the acceleration vector

$$C^1(u, x) := C^*(u, x)R_0C^{'}(u, x)\,,$$

where

$$C^{'}(u, x) := \left[\frac{\partial c_k(u, x)}{\partial u_r},\ k, r = \overline{1, d}\right].$$

The main result of the mentioned paper is the following.

Theorem 5.9. *Let the averaged system* (5.132) *have a twice continuously differentiable Lyapunov function $V(u)$ such that the derivative $\dot V(u)$ along the system satisfies the inequality*

$$\dot V(u) \le -cV(u), \qquad c > 0\,.$$

Let also the velocity $C(u,x)$ be continuously differentiable in u with the derivatives uniformly bounded with respect to $x \in X$.

Moreover, assume that the element-wise majorants for the velocity matrix

$$C^0(u) := \max_x |C^0(u,x)|$$

and for the acceleration vector

$$C^1(u) := \max_x |C^1(u,x)|$$

satisfy the inequality

$$|C^0(u)|V^{''}(u)| + C^{1*}(u)|V^{'}(u)|| \leq cV(u)\,.$$

Then, for $\varepsilon \in (0, \varepsilon_0]$ and a sufficiently small $\varepsilon_0 > 0$, a solution of the initial system (5.131) *is asymptotically stable, that is,*

$$\mathbf{P}\left\{\lim_{t\to\infty} U^\varepsilon(t) = 0\right\} = 1\,.$$

5.6 Averaging for stochastic Ito systems. An asymptotically finite interval

In this section, we will consider questions related to averaging for stochastic Ito systems. As it was mentioned before, one usually studies, for such systems, weak convergence of exact solutions to solutions averaged over finite time intervals as $\varepsilon \to 0$. It is thus interesting to find conditions that would imply a stronger convergence, e.g., mean square convergence, of exact solutions of the stochastic system

$$dx = \varepsilon a(t,x)dt + \sqrt{\varepsilon}b(t,x)dw(t)\,, \tag{5.133}$$

where $w(t)$ is a Wiener process, to solutions of the averaged system

$$dy = \varepsilon a_0(y)dt + \sqrt{\varepsilon}b_0(y)dw(t) \tag{5.134}$$

as $\varepsilon \to 0$.

Similar questions were treated in [49], where a theorem on mean square continuous dependence of solutions of the stochastic system on the parameter has been obtained in the case when the coefficients of the system are integral continuous with respect to the parameter ε. In the deterministic case, a theorem on integral continuity with respect to the parameter yields the first theorem of Bogolyubov that substantiates the averaging method. This is not always the case for stochastic systems.

Indeed, passing to the "slow" time (5.133) transforms it to the system

$$dx = a\left(\frac{\tau}{\varepsilon}, x\right)d\tau + b\left(\frac{\tau}{\varepsilon}, x\right)d\left(\sqrt{\varepsilon}w\left(\frac{\tau}{\varepsilon}\right)\right), \tag{5.135}$$

where the Wiener process also depends on the parameter, whereas there is no such a dependence in [49] but there are only conditions for weak convergence for systems of type (5.135).

It should be remarked that a result close to the first theorem of Bogolyubov was obtained in [193], where it was proved that solutions of system

$$dx = \varepsilon a(t,x)dt + \sqrt{\varphi(\varepsilon)}b(t,x)dw_\varepsilon(t) \tag{5.136}$$

mean square converge as $\varepsilon \to 0$ to solutions of the corresponding averaged system

$$dy = \overline{a}(y)d\tau + \overline{b}(y)dw_0(\tau), \qquad \tau = \varepsilon t\,. \tag{5.137}$$

Also note that there is a theorem proved in [193] on closeness of exact and averaged solutions on the semiaxis in the case where the solution of the averaged system is an equilibrium.

However, these results do not completely cover the problem of substantiation of the averaging method for systems (5.133) and (5.134); the mean square closeness of the corresponding solutions does not follow from these results, which means that this problem needs an additional study.

To this end, let us consider systems (5.133) and (5.134) with the following conditions:

1) the vectors $a(t,x)$ and $b(t,x)$ are continuous in $t \geq 0$, $x \in \mathbf{R}^n$, and are Lipschitz continuous in x with a constant L;

2) the inequalities $|a(t,0)| \leq K$, $|b(t,0)| \leq K$ hold, where K is a constant;

3) the process $w(t)$ is $\mathcal{F}_t$-measurable for every $t \geq 0$, where $\mathcal{F}_t$ is a nondecreasing flow of σ-algebras in the definition of a solution;

4) there exist vectors $a_0(x)$ and $b_0(x)$, and a function $\alpha(T)$ that approach zero as $T \to \infty$ such that

$$\frac{1}{T}\left|\int_0^T [a(t,x) - a_0(x)]dt\right| \leq \alpha(T)(1+|x|),$$

$$\frac{1}{T}\int_0^T |b(t,x) - b_0(x)|^2 dt \leq \alpha(T)(1+|x|^2).$$

We will call system (5.134) with a_0 and b_0 satisfying condition 4) averaged for (5.133) and prove a theorem on closeness of solutions of the exact and averaged systems over intervals of the length order $\frac{1}{\varepsilon}$.

Theorem 5.10. *Let conditions 1)–4) be satisfied. If $x(t)$ and $y(t)$ are solutions of systems (5.133) and (5.134), correspondingly, and $x(0) = y(0) = x_0$, where x_0 is a random variable independent of $w(t)$ and having second moment, then for arbitrary $\eta > 0$ and $T > 0$ there exists $\varepsilon_0 > 0$ such that the following inequality holds for $\varepsilon < \varepsilon_0$:*

$$\mathbf{E}|x(t) - y(t)|^2 < \eta \qquad \textit{for} \qquad t \in \left[0, \frac{T}{\varepsilon}\right]. \tag{5.138}$$

Proof. First of all note that conditions 1) and 4) imply that the functions $a_0(x)$ and $b_0(x)$ are Lipschitz continuous. Then, by the existence and uniqueness theorem for solutions, the Cauchy problems $x(t_0) = x_0$ and $y(t_0) = x_0$ for systems (5.133) and (5.134) have unique strong solutions for $t \geq t_0$.

Using the integral representations for the solutions $x(t)$ and $y(t)$,

$$x(t) = x_0 + \varepsilon \int_0^t a(s, x(s))ds + \sqrt{\varepsilon} \int_0^t b(s, x(s))dw(s)\,, \tag{5.139}$$

$$y(t) = x_0 + \varepsilon \int_0^t a_0(y(s))ds + \sqrt{\varepsilon} \int_0^t b_0(y(s))dw(s)\,, \tag{5.140}$$

and standard estimates for second moments, together with conditions 1) and 2), we get the inequalities

$$E|x(t)|^2 \leq C, \tag{5.141}$$

$$E|y(t)|^2 \leq C \tag{5.142}$$

for $t \in \left[0, \frac{T}{\varepsilon}\right]$, where the constant C depends only on T, K, L, and x_0, and does not depend on ε.

Fix $\eta > 0$ and $T > 0$ and estimate the mean square deviation between the solutions $x(t)$ and $y(t)$ for $t \in \left[0, \frac{T}{\varepsilon}\right]$. It follows from (5.139) and (5.140) that

$$\begin{aligned}
&|x(t) - y(t)| \\
&\quad \leq \varepsilon \left| \int_0^t (a(s, x(s)) - a_0(y(s)))ds \right| + \sqrt{\varepsilon} \left| \int_0^t (b(s, x(s)) - b_0(y(s)))dw(s) \right| \\
&\quad \leq \varepsilon \left| \int_0^t (a(s, x(s)) - a(s, y(s)))ds \right| + \sqrt{\varepsilon} \left| \int_0^t (b(s, x(s)) - b(s, y(s)))dw(s) \right|
\end{aligned}$$

$$+ \varepsilon \left| \int_0^t (a(s, y(s)) - a_0(y(s)))ds \right| + \sqrt{\varepsilon} \left| \int_0^t (b(s, y(s)) - b_0(y(s)))dw(s) \right| .$$

Then

$$\begin{aligned} \mathbf{E}|x(t) - y(t)|^2 \leq 4\varepsilon(TL + L^2) \int_0^t \mathbf{E}|x(s) - y(s)|^2 ds \\ + 4\varepsilon^2 \mathbf{E} \left| \int_0^t (a(s, y(s)) - a_0(y(s)))ds \right|^2 \\ + 4\varepsilon \int_0^t E|b(s, y(s)) - b_0(y(s))|^2 ds . \end{aligned} \tag{5.143}$$

Let us estimate the last two terms in (5.143).

Subdivide the segment $\left[0, \frac{T}{\varepsilon}\right]$ into n parts and find that

$$\begin{aligned} &\mathbf{E} \left| \int_0^t (a(s, y(s)) - a_0(y(s)))ds \right|^2 \\ &\quad \leq \mathbf{E} \left| \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} [(a(s, y(s)) - a(s, y(t_i))) - (a_0(y(s)) - a_0(y(t_i)))]ds \right.\right. \\ &\qquad \left.\left. + \int_{t_i}^{t_{i+1}} [a(s, y(t_i)) - a_0(y(t_i))]ds \right) \right|^2 . \end{aligned}$$

Denote $y_i = y(t_i)$. It follows from the latter inequality that

$$\begin{aligned} &\mathbf{E} \left| \int_0^t (a(s, y(s)) - a_0(y(s)))ds \right|^2 \\ &\quad \leq 2\mathbf{E} \left| \sum_{i=0}^{n-1} \left[\int_{t_i}^{t_{i+1}} [a(s, y(s)) - a(s, y_i)] - [a_0(y(s)) - a_0(y_i)]ds \right] \right|^2 \\ &\qquad + 2\mathbf{E} \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [a(s, y_i) - a_0(y_i)]ds \right|^2 . \end{aligned} \tag{5.144}$$

The first term in this inequality does not exceed the expression

$$\frac{8L^2 T}{\varepsilon} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbf{E}|y(s) - y_i|^2 ds . \tag{5.145}$$

Using the conditions of the theorem and (5.142) we have the following estimate on every segment $[t_i, t_{i+1}]$:

$$\begin{aligned}\mathbf{E}|y(t)-y_i|^2 &\le 2\varepsilon^2\frac{T}{\varepsilon n}\int_{t_i}^{t_{i+1}}\mathbf{E}|a_0(y(s))|^2ds + 2\varepsilon\int_{t_i}^{t_{i+1}}\mathbf{E}|b_0(y(s))|^2ds \\ &\le 2\varepsilon\frac{T}{n}\left(2L^2C+2|a_0(0)|^2\right)\frac{T}{\varepsilon n} + 2\varepsilon\left(2L^2C+2|b_0(0)|^2\right)\frac{T}{\varepsilon n} \le \frac{R}{n},\end{aligned} \tag{5.146}$$

where R is a constant independent of ε and n. Substituting (5.146) into (5.145), we find a final estimate for the first term in (5.144),

$$\begin{aligned}&2\mathbf{E}\varepsilon^2\left|\sum_{i=0}^{n-1}\left[\int_{t_i}^{t_{i+1}}[(a(s,y(s))-a(s,y_i))-(a_0(y(s))-a_0(y_i))]ds\right]\right|^2 \\ &\le \frac{8L^2T\varepsilon^2}{\varepsilon}\sum_{i=0}^{n-1}\frac{R}{n^2}\frac{T}{\varepsilon} = \frac{8L^2T^2R}{n}.\end{aligned} \tag{5.147}$$

Let us estimate the second sum in (5.144). We have

$$2\varepsilon^2\mathbf{E}\left|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}[a(s,y_i)-a_0(y_i)]ds\right|^2 \le 2\varepsilon^2 n\sum_{i=0}^{n-1}\mathbf{E}\left|\int_{t_i}^{t_{i+1}}[a(s,y_i)-a_0(y_i)]ds\right|^2.$$

If t belongs to the segment $[t_i, t_{i+1}],\ i\ge 1$, then using (5.136) we get

$$\begin{aligned}\varepsilon^2\mathbf{E}\left|\int_0^{t_i}[a(s,y_i)-a_0(y_i)]ds\right|^2 &\le \varepsilon^2\frac{T^2}{\varepsilon^2}\alpha^2\left(\frac{T}{\varepsilon n}\right)\mathbf{E}(1+|y_i|)^2 \\ &\le T^2\alpha^2\left(\frac{T}{\varepsilon n}\right)2(1+C).\end{aligned} \tag{5.148}$$

If t lies in the segment $\left[0, \frac{T}{\varepsilon n}\right]$, then

$$\begin{aligned}\varepsilon^2\mathbf{E}\left|\int_0^t[a(s,x_0)-a_0(x_0)]ds\right|^2 &\le \varepsilon^2(t\alpha(t))^2\mathbf{E}(1+|x_0|)^2 \\ &\le (\varepsilon t\alpha(t))^2 2(1+C) = \left(\tau\alpha\left(\frac{\tau}{\varepsilon}\right)\right)^2 2(1+C),\end{aligned} \tag{5.149}$$

where $\tau\in\left[0,\frac{T}{n}\right]$. The expression in the right-hand side of (5.149) tends to zero as $\varepsilon\to 0$ for each fixed τ. Since it is monotone nondecreasing with respect to ε for a fixed τ, by Dini's theorem, it tends to zero as $\varepsilon \to 0$ uniformly in $\tau \in \left[0, \frac{T}{n}\right]$, that is,

$$\sup_{\tau \in \left[0, \frac{T}{n}\right]} \left(\tau\alpha\left(\frac{\tau}{\varepsilon}\right)\right)^2 2(1+C) = F(\varepsilon, n) \to 0, \qquad \varepsilon \to 0\,.$$

This implies that

$$\begin{aligned}
&2\varepsilon^2 \mathbf{E}\left|\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [a(s, y_i) - a_0(y_i)]ds\right|^2 \\
&\quad \le 4n\varepsilon^2 \sum_{i=0}^{n-1} \left(\mathbf{E}\left|\int_0^{t_{i+1}} [a(s, y_i) - a_0(y_i)]ds\right|^2 + \mathbf{E}\left|\int_0^{t_i} [a(s, y_i) - a_0(y_i)]ds\right|^2\right) \\
&\quad \le 16n^2T^2\alpha^2\left(\frac{T}{\varepsilon n}\right)(1+C) + F(\varepsilon, n)\,. \qquad (5.150)
\end{aligned}$$

By choosing a sufficiently large n, the expression in the right-hand side of (5.147) can be made less than

$$\frac{\eta}{4} e^{-4(TL+L^2)T}. \qquad (5.151)$$

Fixing such n and taking a sufficiently small ε we can get an estimate similar to (5.151) for the expression in (5.150).

The third term in (5.143) can be estimated similarly to the second one with the use of condition 4). These estimates and the Gronwall-Bellman lemma yield the inequality

$$\mathbf{E}|x(t) - y(t)|^2 < \eta$$

for $t \in \left[0, \frac{T}{\varepsilon}\right]$, which was to be proved. □

Remark. It follows from the proof of the theorem that the convergence of an exact solution to an averaged one is uniform with respect to the initial conditions x_0 on every ball $\mathbf{E}|x_0|^2 \le R$.

If condition 4) of this theorem is replaced with a stronger condition, namely, the condition that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T a(t,x)dt = a_0(x) \quad \text{and} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T |b(t,x) - b_0(x)|^2 dt = 0$$

uniformly in $x \in \mathbf{R}^n$ with $a_0(x)$ and $b_0(x)$ bounded on $\mathbf{R}$, then convergence in the theorem will be uniform with respect to arbitrary initial conditions.

5.7 Averaging on the semiaxis

In the previous section, we have established the mean square closeness of exact and averaged solutions on asymptotically finite time intervals (of order $\frac{1}{\varepsilon}$). In this section, we will obtain a similar result for the semiaxis $t \geq 0$.

Let us first give the definitions necessary for the sequel.

Definition 5.1. A solution $x(t, t_0, x_0)$ of the stochastic equation

$$dx = f(t, x)dt + g(t, x)dw(t)\,, \tag{5.152}$$

$x(t_0, t_0, x_0) = x_0$, where x_0 is a $\mathcal{F}_{t_0}$-measurable random variable having second moment is called *mean square stable* for $t \geq t_0$ if for arbitrary $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_0)$ such that, if $\mathbf{E}|x_0 - y_0|^2 < \delta$, then

$$\mathbf{E}|x(t, t_0, x_0) - x(t, t_0, y_0)|^2 < \varepsilon \tag{5.153}$$

for $t \geq t_0$, where $x(t, t_0, y_0)$ is a solution of equation (5.152), $x(t_0, t_0, y_0) = y_0$ is $\mathcal{F}_{t_0}$-measurable.

Definition 5.2. A solution $x(t, t_0, x_0)$ is called *mean square uniformly stable* for $t \geq 0$ if it is mean square stable for arbitrary $t_0 \geq 0$ and δ does not depend on t_0.

Definition 5.3. A solution $x(t, t_0, x_0)$ is called *mean square asymptotically stable* for $t \geq t_0$ if it is mean square stable and there exists $\delta_1 = \delta_1(t_0)$ such that, if $\mathbf{E}|x_0 - y_0|^2 < \delta_1$, then

$$\lim_{t\to\infty} \mathbf{E}|x(t, t_0, x_0) - y(t, t_0, y_0)|^2 = 0\,. \tag{5.154}$$

Definition 5.4. A solution $x(t, t_0, x_0)$ is called *mean square uniformly asymptotically stable* for $t \geq 0$ if it is mean square uniformly stable and the limit relation (5.154) holds uniformly in t_0 and y_0.

As before, consider system (5.133) and the averaged system (5.134).

Let system (5.134) have a stationary solution $y = y_0$. Then it is also a stationary solution of the system

$$d\bar{x} = a_0(\bar{x})dt + b_0(\bar{x})dw(t)\,. \tag{5.155}$$

Theorem 5.11. *Let conditions 1)–3) in Theorem 5.10 hold, and let the inequalities*

$$\frac{1}{T}\left|\int_t^{t+T} [a(s, x) - a_0(x)]ds\right| \leq \alpha(T)(1 + |x|)\,, \tag{5.156}$$

$$\frac{1}{T}\int_t^{t+T} |b(s,x)-b_0(x)|^2 ds \le \alpha(T)(1+|x|^2) \tag{5.157}$$

hold uniformly in $t \ge 0$.

If the stationary solution $y = y_0$ *of system* (5.155) *is mean square uniformly asymptotically stable for* $t \ge 0$, *then for arbitrary* $\eta > 0$ *there exist* $\varepsilon_0 = \varepsilon_0(\eta) > 0$ *and* $\delta(\eta) > 0$ *such that*

$$\mathbf{E}|x(t) - y_0|^2 < \eta \qquad \textit{for} \qquad t \ge t_0 \tag{5.158}$$

and $\varepsilon < \varepsilon_0$, *where* $x(t)$ *is a solution of system* (5.133) *for which* $\mathbf{E}|x(t_0) - y_0|^2 < \delta$.

Proof. Let us pass to the "slow" time $\tau = \varepsilon t$ in systems (5.133) and (5.134). Then (5.133) will take the form of (5.135), and (5.134) will become

$$dy = a_0(y)d\tau + b_0(y)d\left(\sqrt{\varepsilon}w\left(\frac{\tau}{\varepsilon}\right)\right). \tag{5.159}$$

Now, fix $\eta > 0$. Without loss of generality, we can assume that $y_0 = 0$ and $t_0 = 0$. Using the uniform asymptotic stability, for a given $\eta > 0$ we can find $\delta = \delta(\eta) > 0$ ($\delta < \eta$) and $T = T(\delta) > 0$ such that if a solution $\bar{x}(t)$ of system (5.155) satisfies the condition

$$\mathbf{E}|\bar{x}(0)|^2 < \delta\,, \tag{5.160}$$

then we have the inequalities

$$\mathbf{E}|\bar{x}(t)|^2 < \frac{\eta}{4} \qquad \text{for} \qquad t \ge t_0\,, \tag{5.161}$$

$$\mathbf{E}|\bar{x}(t)|^2 < \frac{\delta}{4} \qquad \text{for} \qquad t \ge t_0 + T \tag{5.162}$$

for arbitrary $t_0 \ge 0$.

For the chosen $\delta(\eta) > 0$ and $T = T(\delta)$ it follows from Theorem 5.10 that there exists $\varepsilon_0 > 0$ such that

$$\mathbf{E}\left|x\left(\frac{\tau}{\varepsilon}\right) - y_0\left(\frac{\tau}{\varepsilon}\right)\right|^2 < \frac{\delta}{4}, \qquad \tau \in [0,\, T]\,, \tag{5.163}$$

for $\varepsilon < \varepsilon_0$, where $y_0\left(\frac{\tau}{\varepsilon}\right)$ is a solution of system (5.159) with $y_0(0) = x(0)$ and $\mathbf{E}|x(0)|^2 < \delta$.

It follows from the uniform asymptotic stability of solution $\bar{x}_0(\tau)$, $\bar{x}_0(0) = x(0)$, of system (5.155) that

$$\mathbf{E}|\bar{x}_0(\tau)|^2 < \frac{\eta}{4}$$

for $\tau \geq 0$ and

$$\mathbf{E}|\bar{x}_0(T)|^2 < \frac{\delta}{4}\,.$$

Since finite dimensional distributions of the processes $w(t)$ and $\sqrt{\varepsilon} w\left(\frac{t}{\varepsilon}\right)$ coincide for $\varepsilon > 0$, and distributions of the solution $y_0\left(\frac{\tau}{\varepsilon}\right)$ are completely determined by the joint distributions of $y_0(0)$ and $\sqrt{\varepsilon} w\left(\frac{\tau}{\varepsilon}\right)$, using that $y_0(0)$ is independent of $\sqrt{\varepsilon} w\left(\frac{\tau}{\varepsilon}\right)$ and that the solution is unique, we see that the distributions of $y_0\left(\frac{\tau}{\varepsilon}\right)$ coincide with the distributions of $\bar{x}_0(\tau)$ and, hence, their second moments satisfy estimates similar to those for $\bar{x}_0(\tau)$,

$$\mathbf{E}\left|y_0\left(\frac{\tau}{\varepsilon}\right)\right|^2 < \frac{\eta}{4} \qquad \text{for} \qquad \tau \geq 0 \tag{5.164}$$

and

$$\mathbf{E}\left|y_0\left(\frac{T}{\varepsilon}\right)\right|^2 < \frac{\delta}{4}. \tag{5.165}$$

Then (5.163), (5.164), and (5.165) give the estimates

$$\mathbf{E}\left|x\left(\frac{\tau}{\varepsilon}\right)\right|^2 < \eta \qquad \text{for} \qquad \tau \in [0, T] \tag{5.166}$$

and

$$\mathbf{E}\left|x\left(\frac{T}{\varepsilon}\right)\right|^2 < \delta \tag{5.167}$$

for $\tau = T$.

A similar reasoning applied to the segment $[T, 2T]$ gives estimate (5.166) for the solution $x\left(\frac{\tau}{\varepsilon}\right)$, and

$$\mathbf{E}\left|x\left(\frac{2T}{\varepsilon}\right)\right|^2 < \delta$$

for $\tau = 2T$. It is only necessary to take into account here that, since conditions (5.156), (5.157) are uniform in $t \geq 0$, it follows from Remark 1 to Theorem 5.10 that the chosen ε_0 is independent of the points kT and of the initial conditions x_0 such that $E|x_0|^2 \leq \delta$.

Continuing this process for the estimates to the intervals $[kT,\, (k+1)T]$ we finally get estimate (5.158), which finishes the proof of the theorem. $\square$

5.8 The averaging method and two-sided bounded solutions of Ito systems

In this section, we will consider questions related to existence of two-sided solutions of systems (5.133), mean square bounded on the axis, in view of solving the system using the averaging method.

Existence of global two-sided solutions of systems of type (5.133) is a very important nontrivial problem, since they are evolution systems (their solutions can be continued only in one direction). There are very few related results. The monograph [40, p. 202] should be mentioned in this connection, where such a question was studied for stochastic systems with interactions under the assumption that the linear part of the system is exponentially stable, with the nonlinear part being subordinated to the linear one. The monograph also contains references to some other results in this direction.

We will consider system (5.133) assuming that its right-hand side is defined on $\mathbf{R}$, the corresponding Wiener process $w(t)$ is also defined on $\mathbf{R}$, $\mathcal{F}_t$-measurable with respect to a nondecreasing flow of σ-algebras defined on $\mathbf{R}$, and conditions (5.156), (5.157) are satisfied uniformly in $t \in \mathbf{R}$.

Denote $\bar{a}(t,x) = a(t,x) - a_0(x)$, $\bar{b}(t,x) = b(t,x) - b_0(x)$. We will use the following condition:

A) there exist positive numbers M, A, γ such that

$$\int_s^t |\bar{a}(u,x)|^2 du \le A e^{-\gamma t^2}(t-s)(1+|x|^2)\,, \tag{5.168}$$

$$\int_s^t |\bar{b}(u,x)|^2 du \le A e^{-\gamma t^2}(t-s)(1+|x|^2) \tag{5.169}$$

for $s \le t \le -M$.

Theorem 5.12. *Let the conditions of Theorem 5.11 and condition A) hold, and system (5.155) have an equilibrium $y = y_0$, mean square uniformly asymptotically stable for $t_0 \in \mathbf{R}$.*

Then for arbitrary $\eta > 0$ there exists $\varepsilon_0 > 0$ such that system (5.133), *for $\varepsilon < \varepsilon_0$, has a solution $x(t)$ defined on $\mathbf{R}$ for which*

$$\mathbf{E}|x(t) - y_0|^2 < \eta \qquad \textit{for} \qquad t \in \mathbf{R}\,. \tag{5.170}$$

Proof. Without loss of generality, we will again assume that $y_0 = 0$, that is, $a_0(0) = b_0(0) = 0$. Fix arbitrary $\eta > 0$. Using the "slow" time $\tau = \varepsilon t$, as in the proof of Theorem 5.11, we see that for arbitrary $\eta > 0$ there exist $\delta = \delta(\eta) > 0$ and $T = T(\delta) > 0$ such that, if $y\left(\frac{\tau}{\varepsilon}\right)$ is a solution of system (5.159) such that $\mathbf{E}\left|y\left(\frac{\tau_0}{\varepsilon}\right)\right|^2 < \delta$, then $\mathbf{E}\left|y\left(\frac{\tau}{\varepsilon}\right)\right|^2 < \frac{\eta}{2}$ for $\tau \geq \tau_0$, and $\mathbf{E}\left|y\left(\frac{\tau}{\varepsilon}\right)\right|^2 < \frac{\delta}{4}$ for $\tau \geq \tau_0 + T$ with arbitrary $\tau_0 \in \mathbf{R}$ and $\varepsilon > 0$. Therefore, for the usual time t, solution $y(t)$ of system (5.134) such that

$$\mathbf{E}|y(t_0)|^2 < \delta \tag{5.171}$$

satisfies the inequality

$$\mathbf{E}|y(t)|^2 < \frac{\eta}{2} \qquad \text{for} \qquad t \geq t_0\,, \tag{5.172}$$

$$\mathbf{E}|y(t)|^2 < \frac{\delta}{4} \qquad \text{for} \qquad t \geq t_0 + \frac{T}{\varepsilon} \tag{5.173}$$

for arbitrary $t_0 \in \mathbf{R}$ and $\varepsilon > 0$, where T does not depend on t_0 and ε.

Let us partition the left semiaxis $t \leq 0$ with points $-\frac{nT}{\varepsilon}$ (n is an integer) into the segments $\left[-\frac{nT}{\varepsilon}, -(n-1)\frac{T}{\varepsilon}\right]$. Let y_{-T} be an arbitrary $\mathcal{F}_{-\frac{T}{\varepsilon}}$-measurable random variable such that $\mathbf{E}|y_{-T}|^2 \leq \delta$. Consider a solution $x(t)$ of exact system (5.133) satisfying $x\left(-\frac{T}{\varepsilon}\right) = y_{-T}$. A reasoning similar to the one used in Theorem 5.11 easily shows that for a fixed $\eta > 0$ there exists $\varepsilon_1 > 0$ such that the following inequalities hold for $\varepsilon < \varepsilon_1$:

$$\mathbf{E}|x(t)|^2 < \eta \qquad \text{for} \qquad t \in \left[-\frac{T}{\varepsilon}, 0\right],$$

$$\mathbf{E}|x(0)|^2 \leq \frac{\delta}{2}\,.$$

Hence, if $\varepsilon < \varepsilon_0$, then all solutions of the exact system starting in a δ-neighborhood of the point $x = 0$ at $t = -\frac{T}{\varepsilon}$, without leaving its η-neighborhood, enter the $\frac{\delta}{2}$-neighborhood of the point $x = 0$ at $t = 0$.

In view of the mean square uniform asymptotic stability of the zero solution of the averaged system and since it is uniform in $t \in \mathbf{R}$, using conditions (5.156) and (5.157) we similarly see that, if $\varepsilon < \varepsilon_0$, solutions of the exact system, which start in the δ-neighborhood of zero at $t = -\frac{nT}{\varepsilon}$, do not leave the η-neighborhood of zero for $t \in \left[-\frac{nT}{\varepsilon}, -(n-1)\frac{T}{\varepsilon}\right]$ in the sense of the mean square metric, and for $t = -(n-1)\frac{T}{\varepsilon}$, the solutions enter the $\frac{\delta}{2}$-neighborhood of the point $x = 0$ for arbitrary natural n.

Denote by $\mathbf{S}_n(\varepsilon)$ the set of values of solutions of the exact system in the point $t = 0$ such that $t = -\frac{nT}{\varepsilon}$ lies in the δ-neighborhood of zero. By the above and the existence and uniqueness theorem for a solution, this set is not empty for arbitrary natural n and $\varepsilon < \varepsilon_0$. Moreover, we have $\mathbf{S}_n(\varepsilon) \subset \mathbf{S}_{n-1}(\varepsilon)$.

Consider the set

$$\mathbf{S}(\varepsilon) = \bigcap_{n\geq 0} \mathbf{S}_n(\varepsilon)\,.$$

Let us show that it is nonempty. Indeed, since solutions of system (5.133) are mean square continuous with respect to the initial conditions, we have that the points $x(0)$ are interior for the set $\mathbf{S}_n(\varepsilon)$ if $x\left(-\frac{nT}{\varepsilon}\right)$ are interior points of the δ-neighborhood of zero, and the sets $\mathbf{S}_n(\varepsilon)$ are closed.

Denote by $x_n(0)$ the value of a solution $x_n(t)$ of the exact system such that $x_n\left(-\frac{nT}{\varepsilon}\right) = 0$. The point $x_n(0)$ is interior for the set $\mathbf{S}_n(\varepsilon)$.

Let us show that the sequence $\{x_n(0)\}$ is mean square convergent. Using the integral representation for a solution of system (5.133) we have

$$\begin{aligned} |x_n(t) - x_{n-1}(t)| \leq & \left|\varepsilon \int_{-\frac{Tn}{\varepsilon}}^{-\frac{T(n-1)}{\varepsilon}} a(s, x_n(s))ds + \sqrt{\varepsilon}\int_{-\frac{Tn}{\varepsilon}}^{-\frac{T(n-1)}{\varepsilon}} b(s, x_n(s))dw(s)\right| \\ & + \varepsilon\left|\int_{\frac{-n+1}{\varepsilon}T}^{t} [a(s, x_n(s)) - a(s, x_{n-1}(s))]ds\right| \\ & + \sqrt{\varepsilon}\left|\int_{\frac{-n+1}{\varepsilon}T}^{t} [b(s, x_n(s)) - b(s, x_{n-1}(s))]dw(s)\right|. \end{aligned}$$

By passing to second moments in the latter inequality, we get

$$\begin{aligned} & \mathbf{E}|x_n(t) - x_{n-1}(t)|^2 \\ & \quad \leq 3\mathbf{E}\left|\varepsilon \int_{-\frac{Tn}{\varepsilon}}^{\frac{T(-n+1)}{\varepsilon}} a(s, x_n(s))ds + \sqrt{\varepsilon}\int_{-\frac{Tn}{\varepsilon}}^{\frac{T(-n+1)}{\varepsilon}} b(s, x_n(s))dw(s)\right|^2 \\ & \quad + 3\varepsilon(n-1)TL^2 \int_{\frac{-n+1}{\varepsilon}T}^{t} \mathbf{E}|x_n(s) - x_{n-1}(s)|^2 ds \\ & \quad + 3\varepsilon L^2 \int_{\frac{-n+1}{\varepsilon}T}^{t} \mathbf{E}|x_n(s) - x_{n-1}(s)|^2 ds\,, \end{aligned}$$

which using the Gronwall-Bellman lemma shows that

$$\begin{aligned} & \mathbf{E}|x_n(0) - x_{n-1}(0)|^2 \\ & \quad \leq 3\mathbf{E}\left|\varepsilon \int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} a(s, x_n(s))ds + \sqrt{\varepsilon}\int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} b(s, x_n(s))dw(s)\right|^2 \\ & \quad \times e^{3L^2T^2(n-1)^2 + 3L^2T(n-1)}\,. \end{aligned} \tag{5.174}$$

Let us now estimate the first factor in (5.174). We have

$$\mathbf{E}\left|\varepsilon\int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} a(s,x_n(s))ds+\sqrt{\varepsilon}\int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} b(s,x_n(s))dw(s)\right|^2$$

$$\leq 2\varepsilon^2\mathbf{E}\left|\int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} a(s,x_n(s))ds\right|^2+2\varepsilon\int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T}\mathbf{E}|b(s,x_n(s))|^2ds$$

$$\leq 2\varepsilon T\int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T}\mathbf{E}|a(s,x_n(s))|^2ds+2\varepsilon\int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T}\mathbf{E}|b(s,x_n(s))|^2ds\,. \quad (5.175)$$

We now estimate each term in (5.175),

$$2\varepsilon T\int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T}\mathbf{E}|a(s,x_n(s))|^2ds\leq 4\varepsilon T\int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T}\mathbf{E}|\bar{a}(s,x_n(s))|^2ds$$

$$+\,4\varepsilon T\int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T}\mathbf{E}|a_0(x_n(s))|^2ds\,. \quad (5.176)$$

Choose n so large that $(-n+1)\frac{T}{\varepsilon}<-M$. Subdivide the segment $\left[\frac{-nT}{\varepsilon},\frac{-n+1}{\varepsilon}T\right]$ with points t_i into m equal parts and denote $x_i=x_n(t_i)$. Then

$$4\varepsilon T\int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T}\mathbf{E}|\bar{a}(s,x_n(s))|^2ds=4\varepsilon T\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\mathbf{E}[|\bar{a}(s,x_n(s))-\bar{a}(s,x_i)|$$

$$+\,|\bar{a}(s,x_i)|]^2ds\leq 8\varepsilon T\left[\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\mathbf{E}|\bar{a}(s,x_n(s))-\bar{a}(s,x_i)|^2ds\right.$$

$$\left.+\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\mathbf{E}|\bar{a}(s,x_i)|^2ds\right]. \quad (5.177)$$

For the segments $[t_i,t_{i+1}]$, using that $\sup\limits_{t\geq-\frac{n}{\varepsilon}T}\mathbf{E}|x_n(t)|^2\leq\eta$ we get

$$\mathbf{E}|x_n(t)-x_i|^2\leq\mathbf{E}\left|\varepsilon\int_{t_i}^{t} a(s,x_n(s))ds+\sqrt{\varepsilon}\int_{t_i}^{t} b(s,x_n(x))dw(s)\right|^2$$

$$\leq 2\varepsilon^2\frac{T}{\varepsilon m}\int_{t_i}^{t_{i+1}}\left(2L^2\mathbf{E}|x_n(s)|^2+2K^2\right)ds$$

$$+\,2\varepsilon\int_{t_i}^{t_{i+1}}\left(2L^2\mathbf{E}|x_n(s)|^2+2K^2\right)ds$$

$$\leq \frac{2T^2}{m^2}(2L^2\eta + 2K^2) + \frac{2T}{m}(2L^2\eta + 2K^2)\,.$$

Hence, the first sum in (5.177) similarly to (5.147) can be estimated with the quantity

$$\frac{R}{m}, \tag{5.178}$$

where R is a constant independent of ε and m.

To estimate the second sum in (5.177), we use condition A) of the theorem. We have

$$8\varepsilon T \sum_{i=0}^{m-1} \mathbf{E} \int_{t_i}^{t_{i+1}} |\bar{a}(s, x_i)|^2 ds \leq 8\varepsilon T \sum_{i=0}^{m-1} A e^{-\gamma\left(\frac{-n+1}{\varepsilon}T\right)^2} \left(\frac{T}{\varepsilon m}\right)(1 + \mathbf{E}|x_i|^2)$$

$$\leq 8T^2(1+\eta)e^{-\gamma\left(\frac{-n+1}{\varepsilon}T\right)^2}\,. \tag{5.179}$$

For fixed ε and n, we can choose m so large that expression (5.178) becomes smaller than $8T^2(1+\eta)e^{-\gamma\left(\frac{n-1}{\varepsilon}T\right)^2}$. Hence, the first term in inequality (5.176) admits the estimate

$$4\varepsilon T \int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} \mathbf{E}|\bar{a}(s, x_n(s))|^2 ds \leq M_1 e^{\frac{-\gamma T^2}{\varepsilon^2}(n-1)^2}, \tag{5.180}$$

where the constant M_1 does not depend on ε and n.

Let us estimate the second term in inequality (5.175). We have

$$2\varepsilon \int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} \mathbf{E}|b(s, x_n(s))|^2 ds \leq 2\varepsilon \int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} \mathbf{E}[|\bar{b}(s, x_n(s))| + |b_0(x_n(s))|]^2 ds$$

$$\leq 4\varepsilon \int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} \mathbf{E}|\bar{b}(s, x_n(s))|^2 ds + 4\varepsilon \int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} \mathbf{E}|b_0(x_n(s))|^2 ds\,. \tag{5.181}$$

The first term in the latter inequality can be estimated as the first term in inequality (5.176) with the use of inequality (5.169).

Hence, there exists a constant $M_2 > 0$ independent of ε and n such that

$$2\varepsilon T \int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} [\mathbf{E}|\bar{a}(s, x_n(s))|^2 + \mathbf{E}|\bar{b}(s, x_n(s))|^2] ds \leq M_2 e^{-\frac{\gamma T^2}{\varepsilon^2}(n-1)^2}. \tag{5.182}$$

It remains to estimate the second terms in (5.176) and (5.181), that is, the expressions

$$4\varepsilon T \int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} \mathbf{E}|a_0(x_n(s))|^2 ds + 4\varepsilon \int_{-\frac{Tn}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T} \mathbf{E}|b_0(x_n(s))|^2 ds.$$

We will use the following inequalities:

$$
\begin{aligned}
&\mathbf{E}\left[|a_0(x_n(s))|^2+|b_0(x_n(s))|^2\right] \le 2L^2\mathbf{E}|x_n(t)|^2 \\
&= 2L^2\mathbf{E}\left|\varepsilon\int_{-\frac{Tn}{\varepsilon}}^t a(s,x_n(s))ds+\sqrt{\varepsilon}\int_{-\frac{Tn}{\varepsilon}}^t b(s,x_n(s))dw(s)\right|^2 \\
&\le 2L^2\mathbf{E}\left|\varepsilon\int_{-\frac{Tn}{\varepsilon}}^t \bar a(s,x_n(s))ds+\varepsilon\int_{-\frac{Tn}{\varepsilon}}^t a_0(x_n(s))ds+\sqrt{\varepsilon}\int_{-\frac{Tn}{\varepsilon}}^t \bar b(s,x_n(s))dw(s)\right. \\
&\quad\left.+\sqrt{\varepsilon}\int_{-\frac{Tn}{\varepsilon}}^t b_0(x_n(s))dw(s)\right|^2 \le 6L^2\mathbf{E}\left|\varepsilon\int_{-\frac{Tn}{\varepsilon}}^t \bar a(s,x_n(s))ds\right. \\
&\quad\left.+\sqrt{\varepsilon}\int_{-\frac{Tn}{\varepsilon}}^t \bar b(s,x_n(s))dw(s)\right|^2+6L^2\varepsilon^2\frac{T}{\varepsilon}\int_{-\frac{Tn}{\varepsilon}}^t \mathbf{E}|a_0(x_n(s))|^2ds \\
&\quad+6L^2\varepsilon\int_{-\frac{Tn}{\varepsilon}}^t \mathbf{E}|b_0(x_n(s))|^2ds \\
&\le 12L^2\left[\varepsilon^2\frac{T}{\varepsilon}\int_{-\frac{Tn}{\varepsilon}}^t \mathbf{E}|\bar a(s,x_n(s))|^2ds+\varepsilon\int_{-\frac{Tn}{\varepsilon}}^t \mathbf{E}|\bar b(s,x_n(s))|^2ds\right] \\
&\quad+6L^2\varepsilon T\int_{-\frac{Tn}{\varepsilon}}^t \mathbf{E}[|a_0(x_n(s))|^2+|b_0(x_n(s))|^2]ds\,.
\end{aligned}
$$

The last term, by (5.182), does not exceed

$$M_3e^{-\frac{\gamma T^2}{\varepsilon^2}(n-1)^2}+6L^2\varepsilon T\int_{-\frac{Tn}{\varepsilon}}^t \mathbf{E}[|a_0(x_n(s))|^2+|b_0(x_n(s))|^2]ds.$$

If follows from the Gronwall-Bellman inequality that

$$\mathbf{E}[|a_0(x_n(t))|^2+b_0(x_n(t))|^2] \le M_3e^{-\frac{\gamma T^2}{\varepsilon^2}(n-1)^2}e^{6L^2T^2} \tag{5.183}$$

for $t\in\left[-\frac{nT}{\varepsilon},-\frac{(n-1)T}{\varepsilon}\right]$, where M_3 does not depend on ε and n.

Using inequality (5.183) we get that

$$
\begin{aligned}
&4\varepsilon T\int_{-\frac{nT}{\varepsilon}}^{\frac{-n+1}{\varepsilon}T}\mathbf{E}\left[|a_0(x_n(s))|^2+|b_0(x_n(s))|^2\right]ds \\
&\qquad\le 4\varepsilon TM_3e^{-\frac{\gamma T^2(n-1)^2}{\varepsilon^2}}e^{12L^2T}\frac{T}{\varepsilon}=M_4e^{-\frac{\gamma T^2(n-1)^2}{\varepsilon^2}}\,.
\end{aligned} \tag{5.184}
$$

It follows from inequalities (5.174), (5.175), (5.176), (5.180), (5.182), and (5.184) that

$$\mathbf{E}|x_n(0) - x_{n-1}(0)|^2 \leq M_5 e^{-\frac{\gamma T^2(n-1)^2}{\varepsilon^2}} e^{3L^2T^2(n-1)^2+3L^2T(n-1)}. \qquad (5.185)$$

Choose $\varepsilon_2 > 0$ so that $\frac{\gamma}{\varepsilon_2^2} > 3L^2$. Then, for all $\varepsilon \leq \varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$, it follows from (5.185) that the series $x_0(0) + \sum_{k=0}^{\infty} [x_{k+1}(0) - x_k(0)]$ is mean square convergent and, hence, the mean square limit $\lim_{n\to\infty} x_n(0) = x_\infty(0)$ exists. Since the set $\mathbf{S}_n(\varepsilon)$ is closed, it follows that $x_\infty(0) \subset \mathbf{S}_n(\varepsilon)$ for arbitrary n.

Consider now a solution $\tilde{x}(t)$ of system (5.133) such that $\tilde{x}(0) = x_\infty(0)$. Using the structure of the set $\mathbf{S}_n(\varepsilon)$ we see that $\tilde{x}\left(-\frac{nT}{\varepsilon}\right)$ belongs to the δ-neighborhood of the point $x = 0$ for arbitrary natural n. Hence, $\tilde{x}(t)$ can be unlimitedly continued to the left and belongs to the η-neighborhood of the point $x = 0$ for arbitrary $t < 0$.

It immediately follows from Theorem 5.11 that the solution $\tilde{x}(t)$ can be unlimitedly continued to the right and that it belongs to the η-neighborhood of the point $x = 0$, which ends the proof. □

5.9 Comments and References

Section 5.1. Among the methods for analyzing nonlinear dynamical systems, the asymptotic method and the averaging method are particularly important. These methods permit to reduce the investigation of a system with small parameter to a study of an averaged system of a simpler form. For deterministic equations, the procedure and the substantiation of the averaging method is due to M. M. Krylov and M. M. Bogolyubov. Works of M. M. Bogolyubov contain results on closeness of the corresponding solutions of the exact and the averaged systems for both finite and infinite time intervals. In the sequel, the averaging method has been extended in two directions. First, new theorems on closeness of solutions of the corresponding systems were obtained and, second, the averaging method itself has been extended to new classes of equations. An extensive bibliography on the subject is contained, e.g., in the monographs of Mitropol'sky [103] and Khapaev [66].

Already in 1937, M. M. Krylov and M. M. Bogolyubov have demonstrated an effectiveness of applying asymptotic methods of nonlinear mechanics, in particular the averaging method, to a study of impulsive systems. A rigorous substantiation of the method for deterministic impulsive systems was carried out by Samoilenko in [136]. We would also like to mention the work of Trofimchuk [184] where the averaging method has been substantiated for impulsive systems in the case when times of the impulsive effects may have accumulation points. For systems with random impulsive effects, for both fixed and random times, the averaging method has been considered by Tsar'kov and Sverdan [179], Anisimov [4, 5], and others. However, the authors there have made an assumption that the values of the impulses and the times at which they occur have the Markov property, which is not always the case in real problems. The results given in this section have not been published before.

Section 5.2. The main results on the asymptotics of normalized deviations between exact solutions and solutions of the averaged motion for differential equations with random right-hand sides are due to R. Z. Khas'minsky [69, 68], who has proved a limit theorem in a form convenient for applications. The results contained in the latter work has later been obtained with weaker conditions by Borodin in [24]. General theorems on averaging are given in the monograph of Skorokhod [156]. Applications of the averaging method to applied engineering problems are contained in the monograph of Skorokhod, Hoppensteadt, Salehi [159]. The results of this section are contained in authors' work [146].

Section 5.3. Khas'minskii [69, 68], Stratonovich [177], and others have studied small nonlinear oscillations for differential equations with random right-hand sides and small parameter as models by using the averaging method.

Similar studies for deterministic impulsive equations were carried out by Samoilenko and Perestyuk [144]. The results exposed in this section were obtained by the authors in [146].

Section 5.4. This section consists of the results obtained by Stanzhytskij in [161].

Section 5.5. Applications of the averaging method to random dynamical systems with regular perturbations over finite and infinite intervals are treated in the works of Tsar'kov [186, p. 364] and Korolyuk [80], where stability of the initial system was studied in relation to the averaged system. The results discussed in this section were obtained by Martynyuk, Stanzhytskij, and Danilov [99].

Sections 5.6—5.7. For stochastic Ito systems, the averaging method has also turned out to be very useful, — it permits to find the principal term in asymptotic representations of solutions as $\varepsilon \to 0$. This was first found by I. I. Gikhman [49, 51], who discovered that the measures that correspond to solutions of stochastic equations with small parameter are weakly compact. Let us also mention the works of Kolomiets [76, 77], where the averaging method was applied to study oscillation systems perturbed with random "white noise" type forces. The work of Skorokhod [157] is interesting in this regard. Deep results on applications of the averaging method to stochastic functional-differential equations were obtained in the works of E. F. Tsar'kov and his colleagues in [186]. We would also like to mention the works of Makhno [96, 97] on limit behavior of normalized deviations between solutions of the exact and the averaged systems as $\varepsilon \to 0$,see also the works of Buckdahn,Quincampoix, Ouknine [28],J-H Kim [72] and Skorokhod [158] The works cited above mainly deal with the behavior of solutions over finite time intervals. Let us remark that J. Vrkoč in [193] has shown that, under certain conditions, exact and averaged solutions are close on the semiaxis if the averaged solution is an equilibrium. Let us also mention that the above works mainly examine the weak convergence of exact solutions to corresponding averaged solutions over finite intervals, or weak convergence of the normalized deviations, where a linear stochastic differential equation was found for the limit process; this equation is, in fact, an equation in variations for the initial system. The questions that remain to be studied are finding conditions for a stronger, e.g., mean square convergence of exact solutions to the averaged ones. The results in this section were obtained by Samoilenko, Stanzhytskij, Makhmudov in [151].

Section 5.8. The problem of existence of global two-sided solutions of stochastic Ito systems is a very important nontrivial problem, since such systems are evolutionary (their solutions can be continued only in one direction). There are very few results addressing this problem. Let us mention the monograph by Dorogovtsev [40, p. 202], where such a problem is treated for stochastic systems under the condition of coarse exponential stability of the linear part of the system with the nonlinearity being subordinated to the linear part. It also contains references to some other results in this direction. The material exposed in the section is contained in the work of Samoilenko, Stanzhytskij, Makhmudov [151].

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QUALITATIVE AND ASYMPTOTIC ANALYSIS OF DIFFERENTIAL EQUATIONS WITH RANDOM PERTURBATIONS

Differential equations with random perturbations are the mathematical models of real-world processes that cannot be described via deterministic laws, and their evolution depends on random factors. The modern theory of differential equations with random perturbations is on the edge of two mathematical disciplines: random processes and ordinary differential equations. Consequently, the sources of these methods come both from the theory of random processes and from the classic theory of differential equations.

This work focuses on the approach to stochastic equations from the perspective of ordinary differential equations. For this purpose, both asymptotic and qualitative methods which appeared in the classical theory of differential equations and nonlinear mechanics are developed.

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